Optimal Stopping Theory and Opportunistic Transmission Scheduling

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Outline

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Optimal Stopping Theory (OST): It is concerned with the problem of choosing a time to take a given action based on sequentially observed random variables in order to maximize (minimize) an expected payoff (cost).

Examples

1. Maximizing the average in coin tossing problem.
2. House selling Problem.

Optimal Stopping Theory
Opportunistic Scheduling

Introduction

Stopping rule problems are defined by two objects

1. A sequence of random variables, $X_1, X_2, \ldots$, whose joint distribution is assumed to be known.

2. A sequence of real-valued reward functions (may be -ve or even $-\infty$),

$$y_0, y_1(x_1), y_2(x_1, x_2), \ldots, y_\infty(x_1, x_2, \ldots)$$

where,

$$y_0 := \text{reward received if you choose not to take any observation.}$$

$$y_1(x_1) := \text{reward for stopping at 1}\textsuperscript{st}-\text{stage after observing } x_1.$$  

3. **Goal:** To choose a stopping time to maximize the *expected* reward.
A (randomized) stopping rule is a sequence of probabilities of stopping and is represented as,

$$\Phi = (\phi_0, \phi_1(x_1), \phi_2(x_1, x_2), \ldots).$$

Probability of stopping at stage $n$, given that you have observed $X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n$, is given by,

$$0 \leq \phi_n(x_1, \ldots, x_n) \leq 1 \quad \forall \quad n.$$

For non-randomized stopping rules,

$$\phi_n(x_1, \ldots, x_n) = 0 \text{ or } 1 \quad \forall \quad n.$$
Probability Mass Function (pmf) of Stopping Time $N$

The pmf of $N$ given $X = x = (x_1, x_2, \ldots)$ is denoted by,

$$
\Psi = (\psi_0, \psi_1, \psi_2, \ldots, \psi_\infty).
$$

Where

$$
\psi_n(x_1, \ldots, x_n) = P(N = n | X = x) \quad \text{for } n = 0, 1, 2, \ldots
$$

This may be related to stopping rule as follows,

$$
\psi_0 = \phi_0
$$

$$
\psi_1(x_1) = (1 - \phi_0) \phi_1(x_1)
$$

$$
\vdots
$$

$$
\psi_n(x_1, \ldots, x_n) = \left[ \prod_{j=1}^{n-1} (1 - \phi_j(x_1, \ldots, x_j)) \right] \phi_n(x_1 \ldots, x_n)
$$
Problem

Problem, then, is to choose a stopping rule $\Phi$ to maximize the expected return, $V(\Phi)$, given as,

$$V(\Phi) = E[y_N(x_1, \ldots, x_N)]$$

$$V(\Phi) = E\left[\sum_{j=0}^{\infty} \psi_j(x_1, \ldots, x_j) y_j(x_1, \ldots, x_j)\right].$$

"$= \infty$" corresponds to the case when stopping never occurs.
Finite Horizon Problems (FHP) [Ferguson, 2006]

- If it is compulsory to stop after observing $x_1, \ldots, x_T$, we say the problem has horizon $T$
- FHP may be obtained as a special case of the general problem by setting,

\[ y_{T+1} = \ldots = y_\infty = -\infty \]

- Such problems can be solved by method of Backward Induction

Define

\[ V_T^{(T)}(x_1, \ldots x_T) = \max\{y_j(x_1, \ldots, x_j), A\} \]

Where,

\[ A = E \left( V_{j+1}^{(T)}(x_1, \ldots x_j, X_{j+1}) | X_1 = x_1, \ldots, X_j = x_j \right), \]

is the expected return obtained by continuing and using the optimal rule for stages \( j + 1 \) through \( T \), given that we have observed \( X_1 = x_1, \ldots, X_j = x_j \), and

\[ V_j^{(T)}(x_1, \ldots x_j), \]

represents the maximum return one can obtain starting from stage \( J \) and having observed \( X_1 = x_1, \ldots, X_j = x_j \).
Opportunistic Scheduling [Poulakis, 2013]

• Considering basic channel capacity equation

\[ R = W \log_2 \left( 1 + \frac{g.P_{Tx}}{N_o.W} \right) \] (1)

\[ \Rightarrow P_{Tx} \propto \frac{1}{g} \]

• Good channel conditions are explored to get better utilization of energy.

• OST is used to find the optimal time instants, to transmit with minimum energy, depending on channel conditions.

Problem Setup

Figure: Problem setup for single-hop point-to-point wireless link

Assumptions:

- Pdf of channel under consideration is known.
- Transmitter is aware of instantaneous CSI at the receiver.
- $\tau >$ channel coherence time, and $T <$ channel coherence time.
First, we consider that the OTS problem is executed for one time \((E^2OTS) - I\).

The problem is to choose a stopping rule, \(1 \leq N \leq m\), to minimize the expected energy consumption, \(E[N]\), of the device. Where,

\[
E_N = P_N \cdot T + N \cdot E_c = \left( \frac{2^R W}{g_N} - 1 \right) \cdot N_o W T + N \cdot E_c
\]  \(2\)

where, \(E_c = \) energy required for channel measurement

Finite horizon problem with horizon \(D_{max}\).
Multithreshold Policy for $E^2OTS - I$

Using the backward induction to find the optimal stopping rule, we write

$$V_j^{(m)} = \min\{P_j T, A_{m-j}\} + E_c,$$  \hspace{1cm} (3)

where,

$$A_{m-j} = E\left[V_{j+1}^{(m)}((g_1, \ldots, g_j, G_{j+1})|G_1 = g_1, \ldots, G_j = g_j)\right].$$  \hspace{1cm} (4)

Hence, the optimal stopping rule suggests stopping and transmitting at stage $j$ if

$$P_j T \leq A_{m-j}.$$
Optimal Stopping Theory

Opportunistic Scheduling

Multithreshold Policy for $E^2OTS - I$ (contd.)

An average cost of continuing can be considered to be associated with each stage $j$, given as,

$$P_{th,j} = \frac{A_{m-j}}{T} \quad \text{for} \quad j = 0, 1, \ldots, m - 1,$$

$$P_{th,m} = P_{\text{max}} = \frac{A_0}{T}. \quad (5)$$

Using backward induction we can compute $A_{m-j}$ for each individual stage, as following

$$A_{m-j} = E \min \left[ PT, A_{m-j-1} \right] + E_c \quad \text{for} \quad j = 0, \ldots, m - 1,$$

$$= \int_0^{\frac{A_{m-j-1}}{T}} pT \, dF_P + \int_{\frac{A_{m-j-1}}{T}}^{P_{\text{max}}} A_{m-j-1} \, dF_P + E_c \quad (7)$$

where, $F_P(p)$ is $P_{\text{max}}$ normalized cdf of transmission power.
Multithreshold Policy for $E^2OTS - I$ (contd.)

The optimal thresholds associated with each stage $j$, can be calculated as,

$$P_{th,j}^* = \int_0^{P_{th,j}} pdF_P + P_{th,j+1}^* - P_{th,j+1}^* F_P(P_{th,j+1}) + \frac{E_c}{T}$$

(8)

for $j = 0, \ldots, m - 1$, and

$$P_{th,m}^* = \frac{A_0}{T} = P_{max}.$$  

(9)

The policy that minimizes the energy consumption for $E^2OTS - I$ can be given as

if $P_j \leq P_{th,j}^*$ → transmit at $j$

else → postpone
Problem of $E^2OTS - I$ is repeated for $L$ rounds.

\[ \{E_{N_1}, \ldots, E_{N_L}\} \rightarrow \text{Cost Sequence} \]

\[ \{N_1, \ldots, N_L\} \rightarrow \text{Stopping time sequence}. \]

With, $1 \leq N_\ell \leq m$ for $\ell = 1, \ldots, L$.

**Aim:** To minimize the average energy consumption per unit time, i.e. the average power consumption (rate of return).
Average energy consumption per unit time can be expressed as (by law of large nos.)

\[
\frac{\sum_{\ell=1}^{L} EN_{\ell}}{\sum_{\ell=1}^{L} TN_{\ell}} \rightarrow \frac{E[EN]}{E[TN]} \tag{10}
\]

Where,

\[
TN = N\tau + T. \tag{11}
\]

An optimal stopping problem of choosing a stopping rule $1 \leq N \leq m$ to minimize the ratio $\frac{E[EN]}{E[TN]}$. 
Theorem 1

If for some $\lambda$, $\inf_{N \in C} E (E_N - \lambda T_N) = 0$, then
$$\inf_{N \in C} \frac{E[E_N]}{E[T_N]} = \lambda.$$ Moreover, if $\inf_{N \in C} E (E_N - \lambda T_N) = 0$ is attained at $N^* \in C$, then $N^*$ is optimal for minimizing $\inf_{N \in C} \frac{E[E_N]}{E[T_N]}$.

Conversely, if $\inf_{N \in C} \frac{E[E_N]}{E[T_N]} = \lambda$ and if the infimum is attained at at $N^* \in C$, then $\inf_{N \in C} E (E_N - \lambda T_N) = 0$ and the infimum is attained at $N^*$.

$C$ is the class of stopping rules s.t. $C = \{N : N \geq 1, ET_N < \infty\}$
From Theorem 1, following two minimization problems are equivalent

$$\inf_{N \in C} \frac{E[E_N]}{E[T_N]} = \lambda^* \iff \inf_{N \in C} E(E_N - \lambda^* T_N) = 0$$  \hspace{1cm} (12)

The optimal return is given by,

$$V(\lambda) = \inf_{N \in C} [E[E_N] - \lambda E[T_N]] = E[E_{N(\lambda)}] - \lambda E[T_{N(\lambda)}]$$, \hspace{1cm} (13)

where, $N(\lambda)$ is the stopping rule that achieves minimum for $\lambda$.

Optimal rate of return, $\lambda^*$, can be found by solving $V(\lambda^*) = 0$ and hence we can find optimal stopping time $N^* = N(\lambda^*)$. 
Let $Z_N = E_N - \lambda T_N$  
\begin{align*}
&= \left( \frac{2^R W - 1}{g_N} \right) N_o WT + N.E_c - \lambda N \tau - \lambda T \\
&= \left[ \left( \frac{2^R W - 1}{g_N} \right) N_o W - \lambda \right] T + N(E_c - \lambda)
\end{align*}

Given that we have observed $G_1 = g_1, \ldots, G_j = g_j$, the minimum rate of return at stage $j$

$$V_j^{(m)} = \min \{ P_j T - \lambda T, A_{m-j} \} + E_c - \lambda \tau,$$

where,

$$A_{m-j} = E \left[ V_{j+1}^{(m)}((g_1, \ldots, g_j, G_{j+1}) | G_1 = g_1, \ldots, G_j = g_j) \right].$$
\( E^2OTS - II : \) Multithreshold Policy

Hence, the optimal stopping rule suggests stopping and transmitting at stage \( j \) if

\[
P_j T - \lambda T \leq A_{m-j}.
\]

So the transmission power threshold is,

\[
P_{th,j} = \frac{A_{m-j}}{T} + \lambda \quad \text{for} \quad j = 0, 1, \ldots, m - 1, \quad (19)
\]

\[
P_{th,m} = P_{max} = \frac{A_0}{T} + \lambda \quad \text{for} \quad j = m. \quad (20)
\]
Following backward induction, we can compute $A_{m-j}(\lambda)$ for each individual stage,

$$
A_{m-j}(\lambda) = E \min \{P_j T - \lambda T, A_{m-j-1}(\lambda)\} + E_c - \lambda \tau
$$

$$
= \int_0^{A_{m-j-1}(\lambda)} (pT - \lambda T) dF_T + \int_{A_{m-j-1}(\lambda)}^{P_{\text{max}}} A_{m-j-1}(\lambda) dF_T
+ E_c - \lambda \tau \quad \text{for} \quad j = 0, 1, \ldots, m - 1.
$$

Consequently we can compute the corresponding power threshold $P_{th,j}(\lambda)$, for each stage for each $\lambda$, using (19) and (20).
Optimal policy is the collection of thresholds corresponding to optimal rate of return, $\lambda^*$, i.e,

$$P_{th,j}^* = P_{th,j}(\lambda^*) = \frac{A_{m-j}(\lambda^*)}{T} + \lambda^*$$

The policy that minimizes the rate of return for $E^2OTS - II$ is given as,

$$\text{if } P_j \leq P_{th,j}^* \rightarrow \text{transmit at } j$$

$$\text{else } \rightarrow \text{postpone}$$
Proposition 1
Optimal power thresholds $P_{th,j}^*$ are increasing on $j = 1, \ldots, m$ i.e,

$$P_{th,j}^* \leq P_{th,j}^* \quad \text{for} \quad j = 1, \ldots, m - 1.$$ 

**proof:** It is equivalent to show that $A_{i+1}(\lambda^*) \leq A_i(\lambda^*)$, for $i = 0, \ldots, m - 2$. Let $A_1(\lambda^*) > A_0(\lambda^*)$. Then,

$$A_2(\lambda^*) = E \min[PT - \lambda T, A_1(\lambda^*)] + E_c - \lambda \tau$$

$$\geq E \min[PT - \lambda T, A_1(\lambda^*)] + E_c - \lambda \tau = A_1(\lambda^*) > A_0(\lambda^*)$$

Therefore, inductively we have $A_m(\lambda^*) > A_0(\lambda^*) = P_{max}T - \lambda T$. As $A_m(\lambda^*) = 0 \Rightarrow \lambda^* > P_{max}$.

Hence, $A_1(\lambda^*) \leq A_0(\lambda^*)$ and rest of the proof follows similarly by induction.
Proposition 2

\[ V(\lambda^*) = 0 \iff A_m(\lambda^*) = 0 \]

Proposition 3

\( A_j(\lambda) \) is continuous and monotonically decreases as \( \lambda \) increases from 0 to \( +\infty \), \( \forall \), 0, \ldots, \( m \).

For all \( j \), \( A_j(\lambda) \) goes from some positive value (for \( \lambda = 0 \)) to \( -\infty \) (for \( \lambda = \infty \)). Hence, \( A(\lambda) = 0 \) has at least one solution.
More Reading on OTS applications

