Separable Convex Optimization with Linear Ascending Constraints

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Outline

Motivation

Problem Statement

A Distributed Algorithm

Proof Steps

Network Structure

- The network class we consider is analogous to a highway taking traffic to the downtown of a city.

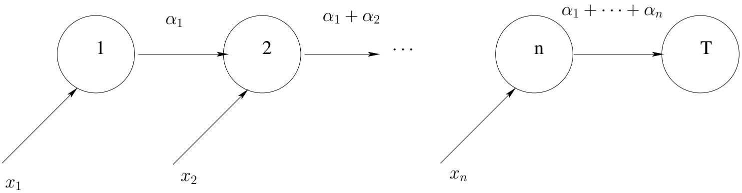

Figure : Network Structure

- n flows across the network.
- Flow i derives a utility $w_i(x_i)$.
- Maximize sum utility subject to the flow constraints of the network.

Problem Statement

$$\text{System}((x_i); W, F):$$

$$\begin{aligned}\text{Maximize} \quad & W(x) := \sum_{i=1}^{n} w_i(x_i) \\ \text{subject to} \quad & x_i \geq 0 \qquad i = 1, 2, \cdots, n, \\ & x_1 \leq \alpha_1, \\ & x_1 + x_2 \leq \alpha_1 + \alpha_2, \\ & \vdots \quad \leq \quad \vdots \\ & x_1 + x_2 + \cdots + x_n = \alpha_1 + \alpha_2 + \cdots + \alpha_n.\end{aligned}$$

- w_i, $i = 1, 2, \cdots, n$ are strictly concave, strictly increasing, continuously differentiable functions.
- $\alpha_i \geq 0$ for $i = 1, 2, \cdots, n$.
- Traffic is elastic (no minimum requirement).

Distributed Optimization

- The network does not know the utility functions.
- Users do not know the network structure.
- Primal ascent and dual descent methods (Arrow-Hurwicz-Uzawa, Low-Lapsley).
- Kelly decomposition
  - Decomposes the system problem into n user problems and a network problem.

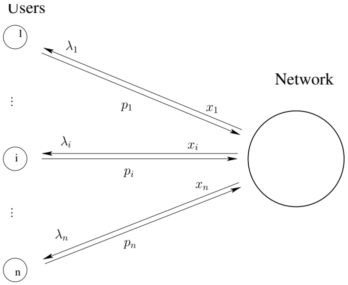

Figure : Kelly Decomposition

Kelly Decomposition

- Choose p_i to maximize the net utility of user i.

$$\text{User}(p_i; \lambda_i): \quad \text{Maximize} \quad w_i\left(\frac{p_i}{\lambda_i}\right) - p_i$$
$$p_i \geq 0.$$

- Based on (p_i), the network allocates rates in a proportionally fair manner.

$$\text{Network}((x_i); (p_i), F): \quad \text{Maximize} \quad \sum_{i=1}^{n} p_i \cdot \log x_i$$
$$(x_i) \in F.$$

- Let (x_i) maximize the network problem. Then $\lambda_i = \frac{p_i}{x_i}$.
- There exists $(\lambda_i^\star), (p_i^\star), (x_i^\star)$ such that
  - $p_i^\star$ solves the user problem for $\lambda_i = \lambda_i^\star$.
  - $(x_i^\star)$ solves the network problem for $(p_i) = (p_i^\star)$ and $x_i^\star = \frac{p_i^\star}{\lambda_i^\star}$.
  - $(x_i^\star)$ solves the system problem.

Kelly Decomposition Contd.

- An iterative method

$$(\lambda_i^{(0)}) \to (p_i^{(0)}) \to (x_i^{(0)}) \underbrace{\to \left(\lambda_i^{(1)} = \frac{p_i^{(0)}}{x_i^{(0)}}\right) \to (p_i^{(1)}) \to (\hat{x}_i)}_{\text{T(x)}}$$

- Restatement of Kelly decomposition: The optimal solution to the system problem, $x^\star$, satisfies

$$x^\star = T(x^\star).$$

- But $T(x)$ has multiple fixed points.

Algorithm and Main Result

- Rate update

$$x^{(k+1)} = (1 - a(k)) \cdot x^{(k)} + a(k) \cdot T(x^{(k)}).$$

- Take $a(k) = \frac{1}{k+1}$.

Theorem

$x^{(k)}$ *converges to* $x^\star$*, the optimal solution to the system problem.*

Proof Steps

- $x^{(k)}$ approximates the trajectory of the following ODE.

$$\dot{x}(t) = T(x) - x.$$

- The equilibrium points of the ODE are the fixed points of $T(x)$.
- $x(t)$ converges to an equilibrium point shown via Lyapunov theory.
    - $W(x)$ is the Lyapunov function.
- The equilibrium point satisfies the KKT conditions of the system problem.
- $x(t) \to x^\star$, hence $x^{(k)} \to x^\star$.

Advantages of Our Algorithm

- Complexity of the algorithm.
  - The network problem can be solved in $\mathcal{O}(n)$ steps using String algorithm (Muckstadt and Sapra).
  - User problem solved in $\mathcal{O}(1)$ steps.
- The Algorithm of Kelly-Maulloo-Tan.
  - The algorithm solves a relaxation of the system problem.
  - It uses Kelly decomposition but does not solve the network problem at each step.
  - Rates allocated at intermediate steps can lie outside the feasible set.
  - This may result in slower convergence to the optimal solution.