Stochastic approximation with set-valued maps and Markov noise: Convergence analysis and applications

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Introduction

Originally conceived as a tool for statistical computation, stochastic approximation schemes today have applications in varied fields of engineering, such as, adaptive signal processing, stochastic optimization, game theory and machine learning to name a few. In control systems engineering too, stochastic approximation is the main paradigm for on-line algorithms for system identification and adaptive control. Stochastic approximation schemes are designed to operate in uncertain environments and are iterative in nature which are the key traits that make them attractive for adaptive schemes.

In our work we investigate the asymptotic behavior of stochastic approximation schemes where the drift function is set-valued depending additionally on an iterate-dependent Markov noise term. Such schemes arise for example, when one is trying to minimize a convex function which is an expectation and is estimated via Markov chain Monte Carlo methods. We adopt the dynamical systems approach to analyze the asymptotic behavior of such schemes. We consider two variants of the same, namely:

1. Single time scale stochastic recursive inclusion (SRI) with Markov noise:
   - Convergence analysis involves showing that the iterates track the flow of a differential inclusion and then invoking the limit-set theorem to characterize the limit sets of the recursion in terms of the dynamics of the differential inclusion.
   - Applications include controlled stochastic approximation, subgradient descent, approximate drift problem and analysis of discontinuous dynamics, all in the presence of Markov noise.

2. Two time scale SRI with Markov noise:
   - Iterates are updated along a slower and faster timescale induced due to a clever choice of step size regimes.
   - Slower timescale iterates appear static w.r.t the faster timescale and faster timescale iterates appear to have equilibrated w.r.t. the slower timescale recursion.
   - Applications include control algorithms in reinforcement learning and solving nested optimization problems, for example: computation of saddle points.

In the analysis above, the iterates are assumed to lie in compact set which is sample path dependent. This stability assumption is essential but is often hard to verify. We all consider the analysis of standard SRI in the absence of a stability guarantee. Our contributions are:

• Extension of the lock-in probability bound to the case with set-valued maps.

• Using the lock-in probability result we show that a feedback mechanism which involves resetting the iterates at regular time intervals, stabilizes the recursion when the mean field possesses a global attractor.

Single timescale SRI with Markov noise [1]

**Recursion and assumptions:** Given \( X_0 \in \mathbb{R}^n, S_0 \subset S \), a compact metric space, \( X_n \)'s are generated according to the recursion

\[
X_{n+1} = X_n - \alpha(n)M_{n+1} \in \mu(n)H(x_n, S_n),
\]

and, \( P_S(n) = AX_S(n), S_n, m = n \subseteq H(x_n, S_n)\), for every \( A, S \subseteq \mathbb{R}^n \) measurable, where,

\( H(\cdot) = \mathbb{R}^n \times S \rightarrow \text{(subsets of } \mathbb{R}^n) \) such that \( H(x,s) \) is non-empty, convex and compact;

\( \mu(\cdot) : \mathbb{R}^n \times S \rightarrow [0, 1] \) is the set-valued map \( H \) has a closed graph.

\( (A) \): \( (S, \mathcal{S}) \) is a separable metric space, \( S \subseteq \mathbb{R}^n \), \( w \in \text{(some constant)} \), \( \mathcal{S} \) satisfies the metric space of probability measures on \( S \) with the metric of weak convergence.

\( (A3) \): \( (\alpha(n), n \geq 0) \) is the step size sequence satisfying the standard Robbins-Monro conditions.

\( (A4) \): \( (\mu(n), n \geq 0) \) denotes the additive noise terms satisfying a condition which ensures that their eventual convergence is negligible over any time interval.

\( (A5) \): The iterates remain stable, i.e., \( P_{\mu(n)}(\cdot | X_0) \subseteq \mathbb{R} \rightarrow [0, 1] \).

**Mean field and its properties:** As a consequence of \( (A2) \) and the compactness of \( S \), the Markov chain with transition probability kernel \( (H(x,s), x, s) \) admits at least one stationary distribution for every \( n \). Let \( \pi(x) \) denote the set of stationary distributions for every \( x \). The differential inclusion (DI) whose flow the iterates generated as in recursion (1) are expected track is given by,

\[
\dot{x} \in -\partial\psi(x) - \partial\phi(x, y, z) \quad x = x(t), \quad t \geq 0.
\]

**Convergence guarantee:** Given that there exists \( x \in \mathbb{R}^n \), a global attractor for the flow of DI (2), then the iterates \( x_n \), as in recursion (1) converge to \( x \) almost surely.

**Applications:**

1. A global attractor is set to which all the solutions of the DI (2) converge to from any initial condition.
2. The main component of the proof of the above convergence result is an asymptotic pseudotrajecto-

**References**


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Borkar\textsuperscript{1} analyzed the iterative scheme of the form

\[ X_{n+1} - X_n - a(n)M_{n+1} = a(n)h(X_n, S_n), \]

where, \( a(n) \) is the step size, \( M_{n+1} \) is the martingale noise, \( h \) is the drift function and \( S_n \) is the Markov noise with \( S_n \in S \), a compact metric space.

\[ \mathbb{P}(S_{n+1} \in A|S_m, X_m, m \leq n) = \Pi(S_n, X_n)(A); \] where \( \Pi : \mathbb{R}^d \times S \to \mathcal{P}(S) \) assumed to be continuous;

\[ \text{Markov chain with TPK } \Pi(\cdot, x)(\cdot) \text{ admits at least one stationary distribution for every } x; D(x). \]

\textsuperscript{1}Vivek S Borkar. “Stochastic approximation with controlled Markov noise”. In: Systems & control letters 55.2 (2006), pp. 139–145.
• Under the additional assumption of stability 
  \( \sup_{n \geq 0} \| X_n \| < \infty \), iterates track the flow of the differential 
  inclusion (DI),
  \[
  \frac{dx}{dt} \in \hat{h}(x),
  \]
  where \( \hat{h}(x) := \bigcup_{\mu \in D(x)} \int_S h(x, s) \mu(ds) \).

• **Convergence guarantee:** If \( x^* \in \mathbb{R}^d \) is globally 
  asymptotically stable for DI (1), then \( X_n \to x^*, \ a.s. \)
Controlled stochastic approximation

\[ X_{n+1} - X_n - a(n)M_{n+1} = a(n)h(X_n, S_n, U_n), \]

where \( U_n \) is the control variable taking value in a compact metric space \( \mathcal{U} \).

Recursion (2) can be equivalently written as,

\[ X_{n+1} - X_n - a(n)M_{n+1} \in a(n)H(X_n, S_n), \]

where, \( H(x, s) := \text{conv} (\{ h(X_n, S_n, u) : u \in \mathcal{U} \}) \).
Subgradient descent:

- Assume that for every $x$, $D(x) = \mu \in \mathcal{P}(S)$.
- Let $J : \mathbb{R}^d \times S \rightarrow \mathbb{R}$ s.t., $J(\cdot, s)$ is convex.
- Consider the recursion given by,

$$X_{n+1} - X_n - a(n)M_{n+1} \in -a(n)\partial J(X_n, S_n),$$

where,

$$\partial J(x, s) := \{g : J(y, s) \geq J(x, s) + \langle g, y - x \rangle, \text{ for every } y\}.$$  

- Does the above recursion minimize the convex function $x \rightarrow \int_S J(x, s)\mu(ds)$?
We analyze the recursion given by,

\[ X_{n+1} - X_n - a(n)M_{n+1} \in a(n)H(X_n, S_n), \]  \hspace{1cm} (3)

where, \( H : \mathbb{R}^d \times S \to \{ \text{subsets of } \mathbb{R}^d \} \) is such that, \( H(x, s) \) is non-empty, convex and compact; \( H \) is upper semicontinuous and \( \|H(x, s)\| \leq K(1 + \|x\|). \)

• Conversion to the single-valued case:

\[ H(x, s) \subseteq H^{(l)}(x, s) \quad \rightarrow \quad \underbrace{h^{(l)}(x, s, u)}_{\text{continuous,}} \ni \big\{ h^{(l)}(x, s, u) : \|u\| \leq 1 \big\} = H^{(l)}(x, s) \]

Using the above, recursion (3) is written as,

\[ X_{n+1} - X_n - a(n)M_{n+1} = a(n)h^{(l)}(X_n, S_n, U_n). \]

• Identifying the limiting trajectory

\[ x^*(t) = \int_0^t \left[ \int_{S \times U} h^{(l)}(x, s, u) \gamma^{(l)}(t; ds, du) \right] dt, \]

where, \( \gamma^{(l)} : [0, \infty) \rightarrow \mathcal{P}(S \times U) \) and for every \( t \geq 0, \gamma_S^{(l)}(t) \in D(x(t)). \)
• The iterates $X_n$ track the flow of the DI$^3$,

$$\frac{dx}{dt} \in \hat{H}(x) := \bigcup_{\mu \in D(x)} \int_{S} H(x, s) \mu(ds).$$

(4)

• Convergence guarantee: If $x^*$ is a global attractor of DI (4), then $X_n \to x^*$ a.s.

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We analyze the recursion given by,

\[ Y_{n+1} - Y_n - b(n)M_{n+1}^{(2)} \in b(n)H_2(X_n, Y_n, S_n^{(2)}), \tag{5a} \]
\[ X_{n+1} - X_n - a(n)M_{n+1}^{(1)} \in a(n)H_1(X_n, Y_n, S_n^{(1)}), \tag{5b} \]

where \( a(n), b(n) \) are step-sizes satisfying \( \lim_{n \to \infty} \frac{b(n)}{a(n)} = 0. \)

• (5a) will appear to be static w.r.t. (5b).
• (5b) will appear to have equilibrated w.r.t (5a).

\[ ^4 \text{Vinayaka Yaji and Shalabh Bhatnagar. “Stochastic Recursive Inclusions in two timescales with non-additive iterate dependent Markov noise”. In: CoRR abs/1611.05961 (2016).} \]
• \( \lambda(y) \) denotes the globally attracting set for the flow of DI,

\[
\frac{dx}{dt} \in \hat{H}_1(x, y)
\]

where, \( \hat{H}_1(x, y) = \bigcup_{\mu \in D^{(1)}(x,y)} \int_{S^{(1)}} H_{1,(x,y)}(s^{(1)}) \mu(ds^{(1)}) \).

• \( \mathcal{Y} \) denotes the globally attracting set for the flow of DI,

\[
\frac{dx}{dt} \in \hat{H}_2(y),
\]

where \( \hat{H}_2(y) := \bigcup_{\mu \in D(y)} \int_{\mathbb{R}^{d_1} \times S^{(2)}} H_{2,y}(x, s^{(2)}) \mu(dx, ds^{(2)}) \), with

\[
D(y) := \left\{ \mu \in \mathcal{P}(\mathbb{R}^{d_1} \times S^{(2)}) : \text{supp}(\mu_{\mathbb{R}^{d_1}}) \subseteq \lambda(y), \right. \\
\left. \mu_{S^{(2)}}(A) = \int_{S^{(2)}} \Pi^{(2)}(x, y, s^{(2)})(A) \mu(dx, ds^{(2)}) \right\}
\]

• **Convergence guarantee:** Under stability,

\((X_n, Y_n) \to \bigcup_{y \in \mathcal{Y}} (\lambda(y) \times \{y\}) \) \ a.s.
We consider the recursion given by,

\[ X_{n+1} - X_n - a(n)M_{n+1} \in a(n)H(X_n) \]

In the absence of a stability guarantee, we show that,

\[ \mathbb{P}(X_n \to A \text{ as } n \to \infty | X_{n_0} \in \mathcal{O}') \geq 1 - 2de^{-\tilde{K}/b(n_0)} \]

where \( A \) is local attractor, \( \mathcal{O}' \) is precompact neighborhood of \( A \) and \( b(n_0) \to 0 \) as \( n_0 \to \infty \).

Using the above we design a feedback mechanism, which stabilizes the scheme in the presence of a global attractor, thus guaranteeing convergence.

Questions ?