Interactive Communication for Data Exchange

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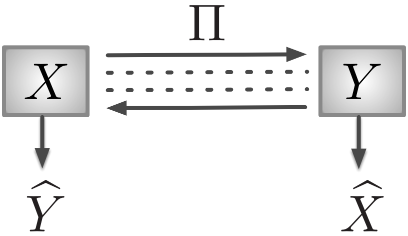
Joint work with Pramod Viswanath and Shun Watanabe

The Data Exchange Problem

[ElGamal-Orlitsky ‘84], [Csiszár-Narayan’04]



A protocol π constitutes an ϵ-data exchange (ϵ-DE) protocol if

$$\Pr\left(\hat{X} = X, \hat{Y} = Y\right) \geq 1 - \epsilon.$$

What is the minimum length $L_\epsilon(X, Y)$ of an ϵ-DE protocol?

The Slepian-Wolf Problem

Only X needs to be sent to an observer of Y.

- [Slepian-Wolf '73] Optimal rate for the case of IID observations:

$$R_\epsilon^* = H(X|Y), \quad 0 < \epsilon < 1.$$

- [Miyake-Kanaya '95] Single-shot bounds:

$$L_\epsilon(X|Y) \geq \lambda + \log\left[1 - \epsilon - \Pr\left(h(X|Y) \leq \lambda\right)\right] : \text{ lower bound}$$

$$L_\epsilon(X|Y) \leq \lambda - \log\left[\epsilon - \Pr\left(h(X|Y) \geq \lambda\right)\right] \quad : \text{ upper bound}$$

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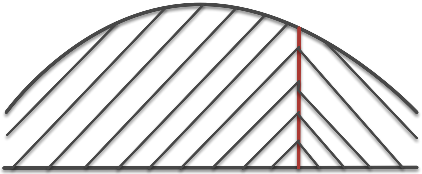

Spectrum of $h(X|Y) = -\log P_{X|Y}(X|Y)$

Can Interaction Help?

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- [Orlitsky '90] Single-shot, worst-case length:

  One round of interaction is almost optimal without error

- [Feder-Shulman'02] Universal version, adaptive rate:

  An interactive protocol accomplishes this task

- [Yang-He '10] Single-shot, average length

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Can interaction help in the data exchange problem?

Using Slepian-Wolf scheme for Data Exchange

The following rate is achievable for the IID case:

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This rate is the least possible.

The proof relies on a property of interactive communication:

$$H(\Pi) \geq H(\Pi|X, U) + H(\Pi|Y, V).$$

Implication: Noninteractive communication can attain the optimal rate

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Is interaction of any use for data exchange?

Main Result: Bounds on $L_\epsilon(X, Y)$

We show that interaction is indeed helpful.

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Define *sum conditional entropy* $h(X \triangle Y) \stackrel{\text{def}}{=} h(X|Y) + h(Y|X)$

Theorem (Single-shot)

For every $0 < \epsilon < 1$, *we have*

$$L_\epsilon(X,Y) \lesssim \lambda - \log\left[\epsilon - \Pr\left(h(X \triangle Y) \geq \lambda\right)\right], \quad \forall\, \lambda > 0,$$

$$L_\epsilon(X,Y) \gtrsim \lambda + \log\left[1 - \epsilon - \Pr\left(h(X \triangle Y) \leq \lambda\right)\right], \quad \forall\, \lambda > 0.$$

Corollary 1: Second-Order Asymptotics for IID Sources

Let $(X^n, Y^n) = (X_i, Y_i)_{i=1}^n$ be IID realizations of (X, Y).

Theorem

For every $0 < \epsilon < 1$, we have

$$L_\epsilon(X^n, Y^n) = nH(X \triangle Y) + \sqrt{n\mathrm{Var}[h(X \triangle Y)]}Q^{-1}(\epsilon) + o(\sqrt{n}).$$

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This length is strictly smaller than that attained by noninteractive protocols.

Corollary 2: Minimum Rate for General Sources

Let $(\mathbf{X}, \mathbf{Y}) = (X_n, Y_n)_{n=1}^{\infty}$ be a general source sequence.

Define the minimum rate of communication for data exchange as

$$R^*(\mathbf{X}, \mathbf{Y}) \stackrel{\text{def}}{=} \inf_{\{\epsilon_n\}} \limsup_n \frac{1}{n} L_{\epsilon_n}(X_n, Y_n),$$

where the infimum is over all sequences $\epsilon_n \to 0$.

Theorem

For a general source sequence $(\mathbf{X}, \mathbf{Y})$*,*

$$R^*(\mathbf{X}, \mathbf{Y}) = \overline{H}(\mathbf{X} \triangle \mathbf{Y}),$$

where $\overline{H}(\mathbf{X} \triangle \mathbf{Y})$ *denotes the* $\limsup$ *in probability of* $h(X_n \triangle Y_n)$*.*

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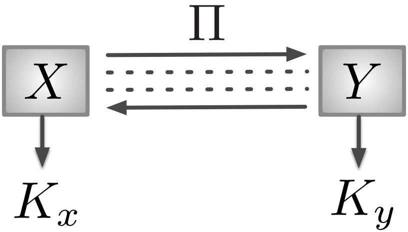
Proof Sketch for the Converse

Digression: Secret Key Agreement



K constitutes an ϵ-secret key of length $\log \mathcal{K}$ if

$$\frac{1}{2}\|\mathrm{P}_{K_x K_y \mathbf{F}} - \mathrm{P}^{(2)}_{\mathtt{unif}} \times \mathrm{P}_{\mathbf{F}}\|_1 \le \epsilon,$$

where

$$\mathrm{P}^{(2)}_{\mathtt{unif}}(k_x, k_y) = \frac{1}{|\mathcal{K}|}\mathbb{1}(k_x = k_y).$$

The maximum length of an ϵ-SK is denoted by $S_\epsilon(X, Y)$.

Main Heuristic

Parties with correlated observations share more bits

than what they communicate.

The extra bits shared can be extracted as a secret key.

Thus, if the parties share $R_{\texttt{shared}}$ bits and communicate R bits,

$$R_{\texttt{shared}} - R \lesssim S(X, Y)$$
$$\Updownarrow$$
$$R_{\texttt{shared}} - S(X, Y) \lesssim R$$

- [Csisár-Narayan '04] First formalized this duality to obtain SK capacity
- [T-Narayan-Gupta '10, T '12] characterization of secure computability

Warm-up: Optimal Rate for Data Exchange

Csiszár-Narayan approach flipped around:

Consider a rate R protocol for data exchange.

- Both parties share roughly $nH(XY)$ bits at the end.
- Using an "extractor lemma" we can generate a SK of rate

$$H(XY) - R,$$

  which must be less than the SK capacity $I(X \wedge Y)$.

Thus,

$$\begin{aligned} R &\geq H(XY) - I(X \wedge Y) \\ &= H(X|Y) + H(Y|X). \end{aligned}$$

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We seek to extend this argument to a single-shot setup.

Upper Bound for Secret Key Length

[T-Watanabe '14]

Theorem

For every $0 < \epsilon, \eta < 1$ *with* $\eta < 1 - \epsilon$*, we have*

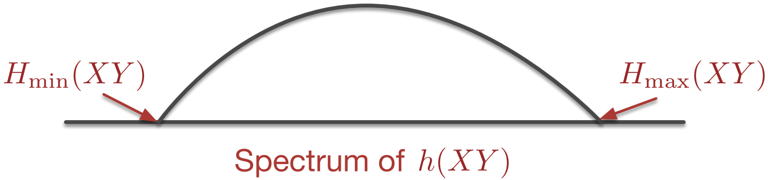
$$S_\epsilon(X, Y) \le \lambda - \log\left(\mathrm{P}_\lambda - \epsilon - \eta\right) + 2\log 1/\eta, \quad \forall\, \lambda > 0,$$

where

$$\mathrm{P}_\lambda = \mathrm{P}_{XY}\left(\left\{(x, y) : \log\frac{\mathrm{P}_{XY}(x, y)}{\mathrm{Q}_X(x)\,\mathrm{Q}_Y(y)} < \lambda\right\}\right).$$

Converse for Almost Uniform Sources

Consider a data exchange protocol of length l.



- Using the *Leftover Hash Lemma*, we can extract a SK of length

$$\approx H_{\min}(XY) - l.$$

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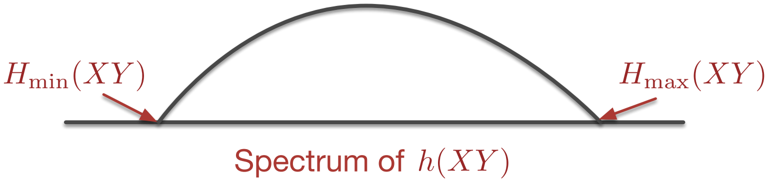

- Using the upper bound for $S_\epsilon(X,Y)$,

$$\begin{aligned} H_{\min}(XY) - l &\lesssim \lambda - \log\left(\mathrm{P}_{XY}\left(i(X \wedge Y) < \lambda\right) - \epsilon\right) \\ &= \lambda - \log\left(\mathrm{P}_{XY}\left(h(XY) - h(X \triangle Y) < \lambda\right) - \epsilon\right) \\ &\leq H_{\max}(XY) - \gamma - \log\left(\mathrm{P}_{XY}\left(h(X \triangle Y) > \gamma\right) - \epsilon\right) \end{aligned}$$

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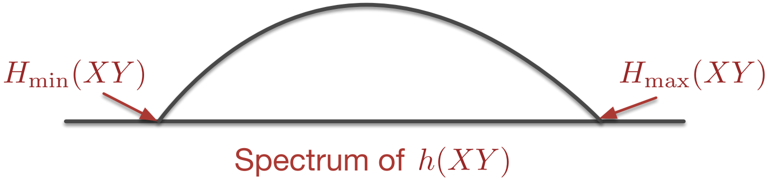


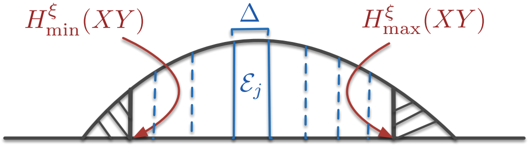
Thus,

$$l \gtrsim H_{\max}(XY) - H_{\min}(XY) + \gamma + \log\left(\mathrm{P}_{XY}\left(h(X \triangle Y) > \gamma\right) - \epsilon\right),$$

which gives the converse bound if $H_{\max}(XY) \approx H_{\min}(XY)$.

General Converse via Spectrum Slicing

Slice the spectrum into N slices of width Δ each.



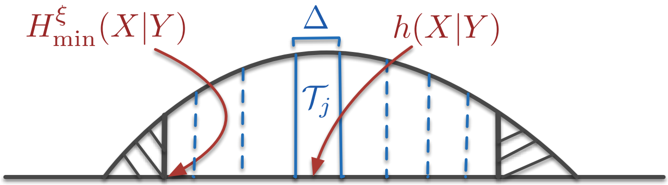
- There exists a slice $\mathcal{E}_j$ with $\mathrm{P}_{XY}(\mathcal{E}_j) \geq N^{-2}$, and so

$$\mathrm{P}_{XY} \leq \mathrm{P}_{XY|\mathcal{E}_j} \leq N^2 \mathrm{P}_{XY}.$$

The proof is completed by applying the previous bound to $\mathrm{P}_{XY|\mathcal{E}_j}$.

Our Achievability Scheme

Rough Sketch of Our Scheme



$$h_i \equiv \begin{cases} \text{random binning of } X \text{ into } H_{\min}^{\xi}(X|Y) \text{ values}, & i = 1, \\ \text{random binning of } X \text{ into } \Delta \text{ values}, & 2 \le i \le N. \end{cases}$$

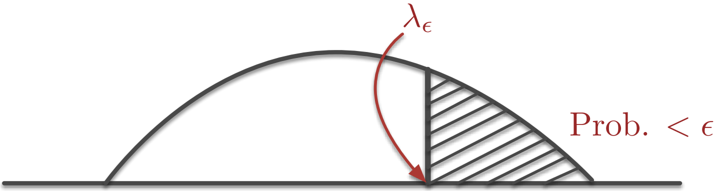
First party sends bin indices $\Pi_i = h_i(x)$ successively until

it receives an ACK or $i = N$

Second party sends an ACK when it finds an $\hat{x}$ s.t.

$$(\hat{x}, y) \in \mathcal{T}_i \quad \text{and} \quad h_j(\hat{x}) = \Pi_j, \quad 1 \le j \le i.$$

In Closing ...



Spectrum of $h(X \triangle Y)$

The minimum length of communication for ϵ-data exchange

is equal to roughly the ϵ-tail λ_ϵ of $h(X \triangle Y)$.

Interaction is necessary to attain this rate.