A Reduced Order Model for Electromagnetic Scattering using Multilevel Krylov Subspace Splitting

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Abstract - Traditional moment matching (Taylor expansion) based reduced order modeling in electromagnetics is known to be narrowband due to ill-conditioned moment generation process. In recent years, multipoint well-conditioned broadband asymptotic waveform evaluation techniques have been introduced that use implicit orthogonalization. These techniques are inherently sequential and difficult to parallelize. This paper introduces an elegant subspace splitting technique that is parallelizable and easy to implement. The technique is shown to match moments and is thus accurate. A wideband scattering problem is used to demonstrate the technique.

I. INTRODUCTION

Numerical techniques, like the finite element method (FEM), are often used to simulate system behavior in various engineering fields like controls, structures, acoustics, etc. [1] [2]. The resulting discreet systems may be large and the solution process may be computationally intensive. Model order reduction (MOR) is used to reduce the size of such large-scale discrete systems. The goal is to capture the behavior of the full system in the reduced order model (ROM) so that the response of the system can be quickly computed for some specified variation in the input parameter. One or more parameters may also be used. For example, in electromagnetic scattering problems, frequency, incidence angle and material parameters are frequently used [3].

One of the MOR techniques, known as the asymptotic waveform evaluation (AWE), is based on matching the Taylor series coefficients (moments). This technique was known to be narrowband due to ill-conditioned moment generation process [4]. However, over the years a few robust and numerically stable methods have been introduced to improve their conditioning [4], [5]. These can be combined with the complex frequency hopping (CFH) [6] approach to cover much larger bandwidths.

The well-conditioned AWE (WCAWE) technique introduced in [4] makes use of correction terms in each iteration to maintain the moment matching property while employing implicit orthogonalization for numerical stability. This excellent technique is, however, cumbersome to implement and difficult to parallelize due to its sequential ROM generation process. The augmented Krylov MOR (AKMOR) technique described in [5] uses a single level subspace splitting approach. While the approach is straightforward to implement, it lacks the moment matching property and will be inaccurate at higher orders compared to WCAWE. In this paper, the AKMOR technique is amended to match moments. As will be seen, this naturally leads to the idea of multilevel splitting. Thus, the new technique will be referred to as Multilevel Krylov subspace splitting MOR (MKMOR). This new approach, not only matches moments, but also is parallelizable and simple to implement.

II. MODEL ORDER REDUCTION

A. Krylov MOR

Fig. 1 illustrates a typical 2D scattering problem. We refer to [5] for the FEM implementation details. The FEM discretization leads to the system

\[ (S + h(s)Z + s^2M)x = b(s), \]

where \( s = j\omega \), \( h(s) = s\sqrt{\varepsilon_0\mu_0} + 1/(2\rho) \) and \( \rho \) is the ABC distance from the origin.

MOR techniques involve projecting (1) onto a lower dimensional subspace. This is done through a congruence transformation

\[ T^*(S + h(s)Z + s^2M)Tx_{red} = Tb(s). \]

where \( x = Tx_{red} \) and \( T \) is the matrix whose columns span the smaller subspace. The reduced system is then given by

\[ (S_{red} + h(s)Z_{red} + s^2M_{red})x_{red} = b_{red}(s). \]

where \( S_{red} = T^*ST \), \( Z_{red} = T^*ZT \), \( M_{red} = T^*MT \) and \( b_{red} = T^*b \), with * denoting conjugate transpose.

In traditional AWE, the columns of \( T \) can be generated using the following iteration:

\[ x_0 = Y_0^{-1}b_0, \]

\[ x_n = \sum_{i=1}^n A_i x_{n-i} + \tau_n, \quad n \geq 1, \]
where
\[ Y_0 = (S + b(s_0)Z + s_0^3M) \]
\[ A_n = -Y_0^{-1}Y_n / n! \]
\[ \tau_n = Y_0^{-1}b_n / n! \], \hspace{1cm} n \geq 1
\]
(5)

Here \( s_0 \) denotes the expansion frequency and the subscripts denote the order of differentiation at \( s_0 \). This process is known to generate an ill-conditioned matrix \( T \). A stable way to generate the moments is described in [4]. We now describe a new multilevel subspace splitting approach.

### B. Multilevel Krylov Subspace Splitting MOR

From (4), a single split MOR can be written as:
\[ x_n = f_0 \]
\[ x_1 = f_1 + \tau_1 \]
\[ x_2 = f_2 + A_1 t_1 + \tau_2 \]
\[ \vdots \]
\[ x_n = f_n + \sum_{i=1}^{n-1} A_i t_{n-i} + \tau_n \]
\[ \text{where} \]
\[ f_0 = Y_0^{-1}b_0, \]
\[ f_n = \sum_{i=1}^{n} A_i f_{n-i}, \quad n \geq 1 \]
\[ t_1 = \tau_1 \]
\[ t_{n-1} = \sum_{i=1}^{n-2} A_i t_{n-i-1} + \tau_{n-1}, \quad n \geq 3 \]

If we let
\[ K_n = [x_0, x_1, \ldots, x_n] \]
\[ K_0^n = [f_0, f_1, \ldots, f_n] \]
\[ \mathcal{W}_i = [\tau_i] \]
\[ \mathcal{V}_i = [A_1 t_1 + \tau_2] \]
\[ \vdots \]
(6)

Then (6) implies that
\[ K_n \in K_0^n + \mathcal{X}_n^1, \]
where \( \mathcal{X}_n^1 = \sum_{i=1}^{n} \mathcal{W}_i \) and the superscripts denote the space id.

It can be observed that \( \mathcal{X}_n^1 \) has the same structure as (4) and has dimension one less than \( K_n \). Therefore, this space can be further split in the same way as above. That is,
\[ K_n \subset K_0^n + \left( \mathcal{X}_n^1 + \mathcal{X}_n^2 \right) \]
(9)

If \( b (\leq n) \) is the maximum number of derivatives of the \( b \) vector to retain, continuing the process \( b \) times, we get:
\[ K_n \subset \mathcal{S} = \sum_{i=0}^{n} K_i^n, \]
(10)

where \( K_i^n \) is an \((n-i+1)\) dimensional Krylov space generated by the ‘seed’ vector \( \tau_i \) and thread \( i \) (see Table 1 below). If \( b = n \), then \( \mathcal{K}_n^b = \mathcal{X}_n^b = [\tau_n] \).

We remark that these Krylov spaces are generated independently of each other. Thus, one can assign each generation process to separate threads/processes in a chosen parallelization framework as visualized in Table 1.

Another point to note is that \( \mathcal{S} \) is a much larger space than the space of first \((n+1)\) moments \( \mathcal{K}_n \). In particular, if \( b = n \), then \( \dim(\mathcal{S}) = (n+1) \times (n+2) / 2 \).

As will be demonstrated in the next section, using this entire space for projection covers a larger bandwidth than what is possible if just \( \mathcal{K}_n \) is used. Nevertheless, since \( \mathcal{K}_n \) is embedded inside \( \mathcal{S} \), it is possible to extract the moment \( x_i \in \mathcal{K}_n \) by simply adding the columns in the \( i \)th row of Table 1. This can be interpreted as a sum reduce operation where all the threads add their contributions to thread 0 (master thread).

<table>
<thead>
<tr>
<th>Iteration</th>
<th>Thread 0</th>
<th>Thread 1</th>
<th>Thread 2</th>
<th>\ldots</th>
<th>Thread min(n,b)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>( f_0 = Y_0^{-1} b_0 )</td>
<td>( f_1' = A_1 f_0' )</td>
<td>( f_1' = Y_0^{-1} b_1 )</td>
<td>( \vdots )</td>
<td>( \vdots )</td>
</tr>
<tr>
<td>1</td>
<td>( f_1' = A_1 f_0' )</td>
<td>( f_2' = A_1 f_1' )</td>
<td>( f_2' = Y_0^{-1} b_2 / 2! )</td>
<td>( \vdots )</td>
<td>( \vdots )</td>
</tr>
<tr>
<td>2</td>
<td>( f_2' = \sum_{i=1}^{2} A_i f_1' )</td>
<td>( f_3' = \sum_{i=1}^{2} A_i f_2' )</td>
<td>( f_3' = \sum_{i=1}^{2} A_i f_2' )</td>
<td>( f_3' = Y_0^{-1} b_3 / 3! )</td>
<td>( \vdots )</td>
</tr>
<tr>
<td>( k )</td>
<td>( f_k' = \sum_{i=1}^{k} A_i f_{k-i} )</td>
<td>( f_{k+1}' = \sum_{i=1}^{k} A_i f_{k-i} )</td>
<td>( f_{k+1}' = \sum_{i=1}^{k} A_i f_{k-i} )</td>
<td>( f_{k+1}' = Y_0^{-1} b_{k+1} / k! )</td>
<td>( \vdots )</td>
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<tr>
<td>( \vdots )</td>
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<td>( \vdots )</td>
<td>( \vdots )</td>
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</tbody>
</table>

TABLE 1. A visual representation of MKMOR. Each thread generates its own Krylov space independently.
Finally the modified Gram-Schmidt method is applied to extract an orthonormal basis for \( \mathcal{K}_n \).

### III. VALIDATION

The reduction technique is applied to the example shown in Fig. 1 where the scatterer is a dielectric cylinder of radius \( a = 0.5 \) m with \( \varepsilon_r = 4 \) and \( \mu_r = 1 \) surrounded by air. The full FEM system required storing three sparse matrices of size 19507x19507. \( \mathbf{S} \) and \( \mathbf{M} \) had 221875 nonzero entries each while \( \mathbf{Z} \) had 11183 nonzero entries.

Fig. 2 compares the radar cross sections (RCS) computed using order 21 ROMs generated using WCAWE and the proposed MKMOR. The results are indistinguishable validating the matching of moments. However, while it took 36.6 s to setup WCAWE ROM, the MKMOR ROM required only 15.6 s even without using parallelization.

Note that the order 21 MKMOR ROM is generated from the moments belonging to \( \mathcal{K}_n \) which is a part of \( \mathcal{S} \). The bandwidth coverage improves significantly if entire space \( \mathcal{S} \) of size 231 (21x22/2) is used. This comes at the cost of bigger ROM size (which is still much smaller than the original FEM system). While a full FEM system solve at any frequency took 0.62 s using the UMFPACK solver, the ROM took 0.25 s. However, the same band can be captured in a more memory efficient way by joining a few small ROMs generated at different expansion points [4][6].

Fig. 3 depicts the error in ROM solution with respect to the FEM solution for WCAWE and MKMOR. As both match moments, the results are indistinguishable for order 21 ROMs but much wider band is covered by MKMOR using full \( \mathcal{S} \) space.

### IV. CONCLUSIONS

A fast and accurate moment matching based MOR technique has been proposed. The technique is elegant in that the original Krylov space structure of traditional AWE method is shown to consist of several independently generated Krylov subspaces. These subspaces can be generated in parallel, although the method is still shown to be nearly 2.5 times faster than another well-conditioned technique when run sequentially. It was shown that the space spanned by the moments is embedded inside the space generated by the technique and can be easily extracted. The method was also shown to be about 2.5 times faster than an efficient sparse direct solver for the full FEM model. This speedup will be more prominent for larger problems, especially 3D.

It can be observed that, for a chosen parallel framework, the approach does not require the threads to communicate and thus belongs to the class of embarrassingly parallel algorithms. However, the threads do not run the same number of iterations. Therefore, a proper load balancing mechanism should improve the efficiency even further. Finally much larger bandwidth should be achievable if parallelized multipoint expansions using CFH are used.

### REFERENCES


