Moment-Matched Lognormal Modeling of Uplink Interference with Power Control and Cell Selection

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***Abstract*—We develop an alternate characterization of the statistical distribution of the inter-cell interference power observed in the uplink of CDMA systems. We show that the lognormal distribution better matches the cumulative distribution and complementary cumulative distribution functions of the uplink interference than the conventionally assumed Gaussian distribution and variants based on it. This is in spite of the fact that many users together contribute to uplink interference, with the number of users and their locations both being random. Our observations hold even in the presence of power control and cell selection, which have hitherto been used to justify the Gaussian distribution approximation. The parameters of the lognormal are obtained by matching moments, for which detailed analytical expressions that incorporate wireless propagation, cellular layout, power control, and cell selection parameters are developed. The moment-matched lognormal model, while not perfect, is an order of magnitude better in modeling the interference power distribution.**

***Index Terms*—lognormal, shadowing, uplink, CDMA, interference, power control, handoff, cell selection.**

I. Introduction

INTERFERENCE plays a crucial role in code division multiple access (CDMA) based cellular communication systems, which use pseudo-random spreading codes for transmitting data. While the spreading codes diminish the interference received from other transmissions, they do not completely annul it. Therefore, cellular system design and analysis requires an accurate statistical characterization of the interference power.

The uplink interference at a base station (BS) is the non-coherent sum of interference signals from the users served by the BS and the users served by other BSs. Consequently, it consists of two components: intra-cell interference and inter-cell interference. While both are random variables, their statistics are vastly different. In the presence of power control, which ensures that the received signal power from each served user is the same, the intra-cell interference power is determined entirely by the number of users served by the base station [1]. On the other hand, the inter-cell interference power is a much more complicated random variable (RV) to characterize because the interfering signals undergo shadowing and fading in their respective wireless channels. Inter-cell interference shall, therefore, be the main focus of this paper.

In the literature, the uplink inter-cell interference power has often been approximated by a Gaussian RV with the same moments, with the central limit theorem being cited as a justification for this [1]. Furthermore, it has been argued that the Gaussian approximation becomes more accurate in the presence of power control and cell selection [2, Chp. 4]. An improvement to the Gaussian approximation based on the Edgeworth approximation, which uses higher order cumulants to modify the moment generation function of the Gaussian distribution, has also been studied [1]. In this paper, we show that lognormal distribution is much more accurate than the Gaussian approximation and its variants in modeling both small and large values of the inter-cell interference even in the presence of power control and cell selection. Such approximations are necessary in the first place because a closed-form expression for the probability distribution of a sum of lognormal RVs is unknown, except for certain special cases [3].

Intuitively, this can be understood as follows. The interfering signal from each user undergoes lognormally distributed shadowing [4]. When the number of interfering signals and their shadowing parameters is deterministically known, it is well understood that the sum of lognormal RVs is well approximated by a lognormal RV [5], [6]. In other words, while the distribution of the sum does eventually become a Gaussian RV, the rate of convergence of the sum distribution to the Gaussian is slow. When the number of summands is the typical number of mobiles in a system, the lognormal distribution better approximates the distribution at both small and large values of interference. Consequently, several methods [3], [7]–[10] have also been proposed for determining the parameters of the approximating lognormal for the case with a fixed number of interferers with a priori specified parameters.

However, it is not a foregone conclusion that the lognormal distribution is a better model for uplink inter-cell interference power. This is because of the following multiple reasons, which pose a new twist to the conventional problem. Firstly, the number of interfering mobiles itself is random in the uplink. Additional randomness is introduced because the transmitting mobiles can be located anywhere within a cell,

Manuscript received April 12, 2009; revised July 27, 2009; accepted September 22, 2009. The associate editor coordinating the review of this letter and approving it for publication was A. Guillen i Fabregas.

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The work by the first, second, and fourth authors was supported in part by the DRDO-IISc Program on Advanced Research in Mathematical Engineering.

A part of this work has appeared in the National Conference on Communications (NCC) 2009, Guwahati, India.

Digital Object Identifier 10.1109/TWC.2010.03.090521

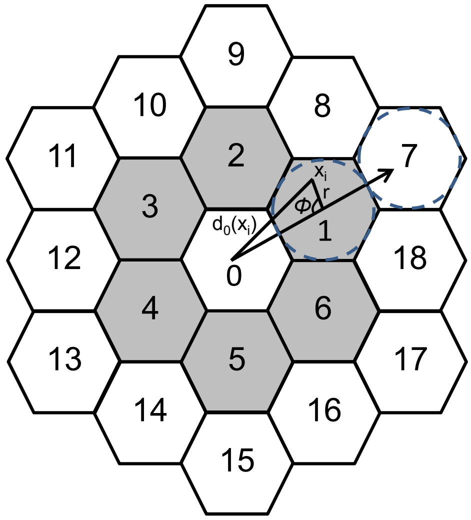

Fig. 1. Hexagonal cellular layout showing reference cell and first- and second-tier cells. Also, shown is the circular cell shape approximation used in the analysis.

which affects the path loss and shadowing contribution to the interference power. The use of power control and cell selection add two additional dimensions to this problem since they affect the power transmitted by the mobiles. Power control compensates for the variable channel gain between the mobile and its *serving BS*. This causes the interference power received by other BSs to be random. Similarly, cell selection affects the inter-cell interference received from a mobile because it determines which BS serves and power controls a mobile.

In this paper, we hypothesize that the uplink inter-cell interference power is better modeled by a lognormal instead of a Gaussian, and verify our hypothesis using several numerical examples. We extend the moment-matching method, first used by Fenton-Wilkinson [7] for the case when the number of interferers is deterministic, to analytically determine the parameters of the approximating lognormal. A key enabler that helps handle the aforementioned additional sources of randomness is the application of spatial Poisson process theory to model the spatial randomness of the interfering mobiles [11]. The elegant theory provides a tractable and practical model for the spatial randomness observed in a CDMA uplink, and has been used effectively in several wireless system design problems [1], [12]. A similar spatial model for user location for computing the cumulants of the uplink interference in the presence of power control and cell selection was also considered in [13]. However, it used three-dimensional Monte Carlo integration for its computations.

We show that while the additional sources of randomness do make the analysis more involved, the simplicity and accuracy of the moment-matching method carry over to a large extent. This holds when only power control is in operation or when both power control and cell selection work together in unison. We derive specific closed-form expressions – in terms of an integral of a simple function – for the first and second order moments of the inter-cell interference power for the cases with (i) power control only, and (ii) with power control and cell selection for hand-off set sizes as large as 3. These demonstrate how the hand-off set size affects the interference. Most importantly, we show that these moments can be used to approximate the probability distribution of the uplink interference by a lognormal. In addition to first-tier and second-tier interferers, our analysis also accounts for 'zero-tier' inter-cell interference that arises in cell selection from users that are geographically located within a cell but are not served by it.

Our results thus provide a more accurate snapshot model for the fading-averaged interference. Consequently, they have applications in cell planning and layout, and, in general, in cellular system design and analysis. It must be noted that while this model is useful, it is not entirely sufficient. For example, to analyze call session or data session specific behavior, a more detailed time trace model or a time correlation model (in addition to the snapshot model studied here) would also be needed. However, this is well beyond the scope of this paper.

The paper is organized as follows. The system model is developed in Sec. II. An analysis of the first- and second-order moments of the interference power and an alternate lognormal model for it is developed in Sec. II-C. Simulation results are presented in Sec. IV, and are followed by our conclusions in Sec. V.

II. System Model

Figure 1 shows the hexagonal cellular layout consisting of a cell 0, which we henceforth refer to as the *reference cell*, and two tiers of interfering cells. A cell k is served by BS k, which is located at the cell's center, with $1 \leq k \leq 6$ for first-tier interferers and $7 \leq k \leq 18$ for second-tier interferers. In the analysis, we shall assume the cells to be circular in shape. Let D denote the distance between BS 0 and a first-tier BS. A second-tier BS is then at a distance of $2D$ from BS 0. We focus on the case when the interference is determined by path loss and shadowing. This case also covers the scenario where Rayleigh or Ricean fading is also present because a composite Rayleigh-lognormal or a Rice-lognormal random variable's distribution can be well approximated by a lognormal [4]. The figure also shows a approximation of the hexagonal cells by circles of radius R, which simplifies the analysis later. For a mobile i inside cell k, let $\mathbf{x}_i$ denote the absolute position vector of mobile i (with respect to a universal center). Let $d_k(\mathbf{x}_i)$ denote the distance of this mobile from BS k.

A. Spatial Poisson Process Model for Users

We first specify the statistics of the number of users in a cell and their locations, as this directly affects the distribution of inter-cell interference. For this, we use the spatial Poisson process model, which provides an analytically tractable and practical model. Briefly, a homogeneous Poisson process is characterized by an intensity parameter λ such that the probability, $P(N_k = l)$, that $N_k = l$ users occur within a cell k of area A is Poisson distributed with mean λA:

$$P(N_k = l) = \frac{(\lambda A)^l}{l!} \exp(-\lambda A). \tag{1}$$

Furthermore, conditioned on N_k, the geographical locations of the N_k mobiles are uniformly distributed over the cell area. We note that the analysis can be extended to handle non-homogeneous Poisson processes as well.

B. Power Control and Cell Selection

When the mobile i transmits a signal with power P_i, the receive signal power at BS k is proportional to $P_i \left(\frac{d_0}{d_k(\mathbf{x}_i)}\right)^\eta s_i^{(k)}$, where d_0 is a reference distance and η is the path loss coefficient, which typically takes values between 2 and 4 [4]. The variable $s_i^{(k)}$ denotes the shadowing of the uplink channel from mobile i to BS k; it is a lognormal random variable (RV), and can be written as

$$s_i^{(k)} = 10^{0.1 y_i(k)} = e^{\beta y_i(k)}, \quad (2)$$

where $y_i(k)$ is a Gaussian RV with zero mean and variance $\sigma_i^2(k)$ and $\beta = 0.1 \log_e(10)$. For analytical simplicity, we assume $\sigma_i^2(k) = \sigma$ for all i and k. Following terminology used in the literature, we shall refer to $\sigma_i^2(k)$ as the dB variance of the lognormal RV $s_i^{(k)}$. Typically, $\sigma_i(k)$ takes values between 4 and 13 [14, Chp. 2.4]. We assume $y_i(k)$ to be independent and identically distributed for different values of i and k.

If k is the serving cell for a user i at location $\mathbf{x}_i$, power control ensures that its transmit power is set so that $P_{\text{tx}} \frac{s_i^{(k)}}{d_k(i)^\eta} = \gamma$, where γ is the power control target [4, Chp. 18]. Hence, the interference, $R(\mathbf{x}_i, k)$, from the user i at BS 0 is

$$R(\mathbf{x}_i, k) = P_{\text{tx}} \frac{s_i^{(0)}}{d_0(\mathbf{x}_i)^\eta} = \gamma e^{\beta(y_i(0) - y_i(k))} \left(\frac{d_k(\mathbf{x}_i)}{d_0(\mathbf{x}_i)}\right)^\eta. \quad (3)$$

Therefore, the total interference at BS 0 from N_k users served by cell k equals $\sum_{i=1}^{N_k} R(\mathbf{x}_i, k)$, which is equal to $\gamma \sum_{i=1}^{N_k} e^{\beta(y_i(0)-y_i(k))} \left(\frac{d_k(\mathbf{x}_i)}{d_0(\mathbf{x}_i)}\right)^\eta$.

With cell selection, a user at $\mathbf{x}_i$ located in cell k maintains a finite sized list called a *handoff set*, $\Gamma_{\mathbf{x}_i}$, of $|\Gamma_{\mathbf{x}_i}|$ geographically nearest BSs, and chooses the one with the strongest channel. The choice depends on both path loss and shadowing. As the notation suggests, the handoff set depends on the user location. Different users at different locations within a cell can have different handoff sets depending on the size allowed for the handoff set and which BSs are closest to them. (This is characterized in detail in Sec. III-C and Sec. III-D.)

Note: The spatial Poisson model together with power control implies that the intra-cell interference power received by any cell of area A from the users it serves is a Poisson random variable with mean $\gamma\lambda A$ [1]. Therefore, as argued before, only the probability distribution of the inter-cell interference remains to be characterized, as is done below.

C. Moment Method Approximation for Interference Power

We shall determine the two parameters of the approximating lognormal RV by matching its first two moments. Let μ_R and σ_R^2 denote the mean and variance, respectively, of the total interference at reference cell 0. Then the parameters of the approximating lognormal, after matching the first two moments, are given as [7]:

$$\mu_{\text{eq}} = \log\left(\frac{\mu_R^2}{\sqrt{\sigma_R^2 + \mu_R^2}}\right), \quad (4)$$

$$\sigma_{\text{eq}} = \sqrt{\log\left(\frac{\sigma_R^2}{\mu_R^2} + 1\right)}. \quad (5)$$

In our case, the first two moments of the interference power from all cells (indexed by k) are

$$\mu_R = \sum_k \mathbf{E}\left[\sum_{N_k=0}^{\infty} \sum_{i=1}^{N_k} R(\mathbf{x}_i, k) e^{-\lambda A} \frac{(\lambda A)^{N_k}}{N_k!}\right], \\ = \sum_k \lambda A \mathbf{E}\left[R(\mathbf{x}_i, k)\right], \quad (6)$$

and

$$\sigma_R^2 = \sum_k \lambda A \left(\mathbf{E}\left[R(\mathbf{x}_i, k)^2\right] - \mathbf{E}\left[R(\mathbf{x}_i, k)\right]^2\right), \quad (7)$$

where $\mathbf{E}[.]$ denotes expectation. The above two equations show that the moment-matching method deals with the randomness in the number of users only through its first two moments. The main task at hand is to evaluate $\mathbf{E}[R(\mathbf{x}_i, k)^m]$, $m = 1, 2$, where the expectation is first over shadowing and then over $\mathbf{x}_i$.

We first consider uplink interference power with power control. Thereafter, we also incorporate cell selection into our analysis. This two step approach will enable us to gauge the relative impact of power control and cell selection on the interference model.

III. Uplink Inter-Cell Interference Power Moments

A. With Power Control

With only power control, a user that is located within cell k is served by BS k. From (3), the mth moment ($m = 1, 2$) of interference power from user i served by BS k, when averaged over both shadowing and user location, $\mathbf{x}_i$, equals

$$\mathbf{E}\left[R(\mathbf{x}_i, k)^m\right] = \gamma^m \mathbf{E}\left[e^{m\beta(y_i(0)-y_i(k))}\right] \mathbf{E}\left[\left(\frac{d_k(\mathbf{x}_i)}{d_0(\mathbf{x}_i)}\right)^{m\eta}\right]. \quad (8)$$

Therefore,

$$\mathbf{E}\left[R(\mathbf{x}_i, k)^m\right] = \gamma^m e^{m^2\beta^2\sigma^2} \\ \times \int_0^R \int_0^{2\pi} \frac{r}{\pi R^2} \left(1 + \frac{D_k^2}{r^2} - 2\frac{D_k}{r}\cos\phi\right)^{-m\eta/2} d\phi \, dr, \quad (9)$$

where $D_k = D$ or $2D$ depending on whether cell k is a first-tier or second-tier cell. Using the variable substitutions $u = \frac{D_k}{r}$ and $w = \cos\phi$, and employing Gauss-Chebyshev quadrature [15] yields

$$\mathbf{E}\left[R(\mathbf{x}_i, k)^m\right] = \gamma^m e^{m^2\beta^2\sigma^2} \frac{2}{W} \left(\frac{D_k}{R}\right)^2 \\ \times \sum_{n=1}^{W} \int_{\frac{D_k}{R}}^{\infty} \frac{1}{u^3} \left(1 + u^2 - 2ua_n'\right)^{-m\eta/2} du, \quad (10)$$

where W is the number of quadrature terms and a_n', $1 \le n \le W$, are the abscissa of Gauss-Chebyshev quadrature. For our method, $W = 6$ turned out to be sufficiently accurate.

B. With Power Control and Cell Selection

With cell selection, a user i located in cell k need not be served by it. Also, the moments of interference power now depend on whether the reference cell 0 belongs to the handoff set of a user located at $\mathbf{x}_i$. This is because when $0 \in \Gamma_{\mathbf{x}_i}$, cell 0 can itself become the serving cell, in which case the user does not cause inter-cell interference to cell 0.

Case 1: Reference Cell in Handoff Set ($0 \in \Gamma_{\mathbf{x}_i}$):

When $0 \in \Gamma_{\mathbf{x}_i}$, the inter-cell interference power at BS 0 from a user at $\mathbf{x}_i$ *located in cell* k is

$$R(\mathbf{x}_i, k) = \gamma e^{\beta(y_0 - M_0(\mathbf{x}_i))} \min_{j \in \Gamma_{\mathbf{x}_i}} e^{-\beta(y_j - M_j(\mathbf{x}_i))}, \quad (11)$$

if $y_0 - M_0(\mathbf{x}_i) < y_j - M_j(\mathbf{x}_i)$ for some $j \in \Gamma_{\mathbf{x}_i} \setminus \{0\}$. Otherwise, $R(\mathbf{x}_i, k)$ is 0. Here, $M_j(\mathbf{x}_i) = \eta\xi \log d_j(\mathbf{x}_i)$.

Let $\mathcal{A}_{\text{out}}$ denote the region of cell k in which this is true. Therefore, conditioned on $\mathbf{x}_i \in \mathcal{A}_{\text{out}}$, the mth moment of the interference power equals

$$\begin{aligned}
&\mathbf{E}\left[R(\mathbf{x}_i,k)^m | \mathbf{x}_i \in \mathcal{A}_{\text{out}}, \mathbf{x}_i\right] \\
&\quad = \frac{\gamma^m}{2\pi\sigma^2} \sum_{l \in \Gamma_{\mathbf{x}_i} \setminus \{0\}} \int_{-\infty}^{\infty} e^{\frac{-y_l^2}{2\sigma^2}} e^{m\beta(M_l(\mathbf{x}_i) - y_l)} \\
&\quad \times \int_{-\infty}^{y_l - M_l(\mathbf{x}_i) + M_0(\mathbf{x}_i)} e^{m\beta(y_0 - M_0(\mathbf{x}_i))} e^{\frac{-y_0^2}{2\sigma^2}} \\
&\times \left[\prod_{j \in \Gamma_{\mathbf{x}_i} \setminus \{l,0\}} Q\left(\frac{M_l(\mathbf{x}_i) - M_j(\mathbf{x}_i) - y_l}{\sigma} \right) \right] dy_0 \, dy_l,
\end{aligned} \quad (12)$$

where $\mathbf{E}[X|Y]$ denotes the expectation of X given Y. The above expression is obtained by conditioning on a cell $l \in \Gamma_{\mathbf{x}_i} \setminus \{0\}$ being the serving cell; $\prod_{j \in \Gamma_{\mathbf{x}_i} \setminus \{l,0\}} Q\left(\frac{M_l(\mathbf{x}_i) - M_j(\mathbf{x}_i) - y_l}{\sigma}\right)$ is the probability that this happens. Using Gauss-Hermite quadrature [15] for the integral over y_l, we get

$$\begin{aligned}
&\mathbf{E}\left[R(\mathbf{x}_i,k)^m | \mathbf{x}_i \in \mathcal{A}_{\text{out}}, \mathbf{x}_i\right] \\
&\approx \frac{\gamma^m}{\pi\sigma\sqrt{2}} \sum_{l \in \Gamma_{\mathbf{x}_i} \setminus \{0\}} \sum_{n=1}^{W} w_n e^{m\beta\left(M_l(\mathbf{x}_i) - M_0(\mathbf{x}_i) - \sqrt{2}\sigma a_n\right)} \\
&\quad \times \left[\prod_{j \in \Gamma_{\mathbf{x}_i} \setminus \{l,0\}} Q\left(\frac{M_l(\mathbf{x}_i) - M_j(\mathbf{x}_i) - \sqrt{2}\sigma a_n}{\sigma} \right) \right] \\
&\quad\quad \times \int_{-\infty}^{\sqrt{2}\sigma a_n - M_l(\mathbf{x}_i) + M_0(\mathbf{x}_i)} e^{m\beta y_0 - \frac{y_0^2}{2\sigma^2}} \, dy_0,
\end{aligned} \quad (13)$$

where a_n and w_n, $1 \leq n \leq W$, are the quadrature abscissa and weights that are readily tabulated in [15].[1] The inner integral evaluates to

$$\begin{aligned}
&\mathbf{E}\left[R(\mathbf{x}_i,k)^m | \mathbf{x}_i \in \mathcal{A}_{\text{out}}, \mathbf{x}_i\right] \\
&\approx \frac{\gamma^m}{\sqrt{\pi}} \sum_{l \in \Gamma_{\mathbf{x}_i} \setminus \{0\}} \sum_{n=1}^{W} w_n e^{m\beta\left(M_l(\mathbf{x}_i) - M_0(\mathbf{x}_i) - \sqrt{2}\sigma a_n\right) + m^2\beta^2/2} \\
&\quad \times \left[\prod_{j \in \Gamma_{\mathbf{x}_i} \setminus \{l,0\}} Q\left(\frac{M_l(\mathbf{x}_i) - M_j(\mathbf{x}_i) - \sqrt{2}\sigma a_n}{\sigma} \right) \right] \\
&\quad \times Q\left(\frac{M_l(\mathbf{x}_i) - M_0(\mathbf{x}_i) - \sqrt{2}\sigma a_n + m\beta\sigma^2}{\sigma} \right).
\end{aligned} \quad (14)$$

Case 2: Reference Cell Not in $\Gamma_{\mathbf{x}_i}$ ($0 \notin \Gamma_{\mathbf{x}_i}$):

When $0 \notin \Gamma_{\mathbf{x}_i}$, we have

$$R(\mathbf{x}_i, k) = \gamma e^{\beta(y_0 - M_0(\mathbf{x}_i))} \min_{l \in \Gamma_{\mathbf{x}_i}} e^{-\beta(y_l - M_l(\mathbf{x}_i))}. \quad (15)$$

Let $\mathcal{A}_{\text{in}}$ denote the region of cell k in which this is true. The mth moments of the inter-cell interference conditioned on $\mathbf{x}_i$ are then given by

$$\begin{aligned}
&\mathbf{E}\left[R(\mathbf{x}_i,k)^m | \mathbf{x}_i \in \mathcal{A}_{\text{in}}, \mathbf{x}_i\right] = \gamma^m e^{-m\beta M_0(\mathbf{x}_i)} \mathbf{E}\left[e^{m\beta y_0}\right] \\
&\times \sum_{l=1}^{|\Gamma_{\mathbf{x}_i}|} \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[\prod_{j \in \Gamma_{\mathbf{x}_i} \setminus \{l\}} Q\left(\frac{M_l(\mathbf{x}_i) - M_j(\mathbf{x}_i) - y_l}{\sigma} \right) \right] \\
&\quad \times e^{m\beta(M_l(\mathbf{x}_i) - y_l)} e^{-y_l^2/2\sigma^2} \, dy_l.
\end{aligned} \quad (16)$$

The above expression is obtained, as before, by conditioning on a cell $k \in \Gamma_{\mathbf{x}_i} \setminus \{0\}$ being the serving cell, which happens with probability $\prod_{j \in \Gamma_{\mathbf{x}_i} \setminus \{l\}} Q\left(\frac{M_l(\mathbf{x}_i) - M_j(\mathbf{x}_i) - y_l}{\sigma}\right)$.

Using appropriate variable substitutions followed by Gauss-Hermite quadrature, we get

$$\begin{aligned}
&\mathbf{E}\left[R(\mathbf{x}_i,k)^m | \mathbf{x}_i \in \mathcal{A}_{\text{in}}, \mathbf{x}_i\right] \\
&\approx \frac{\gamma^m}{\sqrt{\pi}} e^{-m\beta M_0(\mathbf{x}_i)} e^{m^2\beta^2\sigma^2/2} \sum_{l=1}^{|\Gamma_{\mathbf{x}_i}|} e^{m\beta M_l(\mathbf{x}_i)} \left[\sum_{n=1}^{W} w_n \right. \\
&\quad \left. \times \prod_{j \in \Gamma_{\mathbf{x}_i} \setminus \{l\}} Q\left(\frac{M_l(\mathbf{x}_i) + \sqrt{2}\sigma a_n + m\beta\sigma^2 - M_j(\mathbf{x}_i)}{\sigma} \right) \right].
\end{aligned} \quad (17)$$

Therefore, the spatial user distribution averaged moments of interference from cell k are

$$\begin{aligned}
&\mathbf{E}\left[R(\mathbf{x}_i,k)^m\right] \\
&\quad = \frac{1}{\pi R^2} \left(\int_{\mathcal{A}_{\text{out}}} r \mathbf{E}\left[R(\mathbf{x}_i,k)^m | \mathbf{x}_i \in \mathcal{A}_{\text{out}}, \mathbf{x}_i\right] dr d\phi \right. \\
&\quad\quad \left. + \int_{\mathcal{A}_{\text{in}}} r \mathbf{E}\left[R(\mathbf{x}_i,k)^m | \mathbf{x}_i \in \mathcal{A}_{\text{in}}, \mathbf{x}_i\right] dr d\phi \right),
\end{aligned} \quad (18)$$

where (r, ϕ) are the polar coordinates of $\mathbf{x}_i$ with respect to BS k.

The last step in the analysis is specifying $\mathcal{A}_{\text{out}}$ and $\mathcal{A}_{\text{in}}$, and the distances from the serving and reference cell centers that determine $M_k(\mathbf{x}_i)$ in (14) and (17). This depends on the handoff set size and on the relative location of the interfering cell. The cases are enumerated below. Since the interference

[1] $\mathbf{E}\left[R(\mathbf{x}_i,k)^m | \mathbf{x}_i \in \mathcal{A}_{\text{out}}, \mathbf{x}_i\right]$ can, in fact, be shown to exactly equal $\gamma^m e^{m\beta(M_1(\mathbf{x}_i) - M_0(\mathbf{x}_i)) + m^2\beta^2\sigma^2} Q\left(\frac{M_1(\mathbf{x}_i) + 2m\beta\sigma^2 - M_0(\mathbf{x}_i)}{\sigma\sqrt{2}}\right)$ when $|\Gamma_{\mathbf{x}_i}| = 2$. However, we do not use it as it does not generalize for $|\Gamma_{\mathbf{x}_i}| > 2$.

from the six first-tier cells (or the twelve second-tier cells) is statistically identical and independent, it suffices to describe the analysis for a specific first-tier cell (e.g., 1) and a specific second-tier cell (e.g., 7). With cell selection, note that even a user located inside the reference cell can cause inter-cell interference when it happens to choose a neighboring cell as its serving cell. As before, (r, ϕ) will denote the polar coordinates of user location $\mathbf{x}_i$ with respect to the center of cell it is located in.

C. Cell Selection With $|\Gamma_{\mathbf{x}_i}| = 2$ Cells

1) Interference from First-Tier Cell 1: As shown in Fig. 1, we have $d_1(\mathbf{x}_i) = r$ and $d_0(\mathbf{x}_i) = \sqrt{D^2 + r^2 - 2rD\cos\phi}$.

For $-\frac{\pi}{6} \le \phi < \frac{\pi}{6}$, we have $\Gamma_{\mathbf{x}_i} = \{1, 0\}$. In the remaining cases below, BS 0 is never in $\Gamma_{\mathbf{x}_i}$.

For $\frac{\pi}{6} \le \phi < \frac{\pi}{2}$, we have $\Gamma_{\mathbf{x}_i} = \{1, 2\}$ and $d_2(\mathbf{x}_i) = \sqrt{D^2 + r^2 - 2rD\cos\left(\phi - \frac{\pi}{3}\right)}$. For $\frac{\pi}{2} \le \phi < \frac{5\pi}{6}$, we have $\Gamma_{\mathbf{x}_i} = \{1, 8\}$ and $d_8(\mathbf{x}_i) = \sqrt{D^2 + r^2 - 2rD\cos\left(\phi - \frac{2\pi}{3}\right)}$. For $\frac{5\pi}{6} \le \phi < \frac{7\pi}{6}$, we have $\Gamma_{\mathbf{x}_i} = \{1, 7\}$ and $d_7(\mathbf{x}_i) = \sqrt{D^2 + r^2 - 2rD\cos(\phi - \pi)}$. For $\frac{7\pi}{6} \le \phi < \frac{3\pi}{2}$, we have $\Gamma_{\mathbf{x}_i} = \{1, 18\}$ and $d_{18}(\mathbf{x}_i) = \sqrt{D^2 + r^2 - 2rD\cos\left(\phi - \frac{4\pi}{3}\right)}$. For $\frac{3\pi}{2} \le \phi < \frac{11\pi}{6}$, we have $\Gamma_{\mathbf{x}_i} = \{1, 6\}$ and $d_6(\mathbf{x}_i) = \sqrt{D^2 + r^2 - 2rD\cos\left(\phi - \frac{5\pi}{3}\right)}$.

2) Interference from Second-Tier Cell 7: In this case, $0 \notin \Gamma_{\mathbf{x}_i}$ as cell 0 is never one of the closest two cell sites for a user in a second-tier cell. Thus, $d_0(\mathbf{x}_i) = \sqrt{4D^2 + r^2 - 4rD\cos\phi}$ and $d_7(\mathbf{x}_i) = r$. For cell 7, the distance from the second BS in $\Gamma_{\mathbf{x}_i}$ can be compactly stated as: $d_m(\mathbf{x}_i) = \sqrt{D^2 + r^2 - 2rD\cos\left(\left\lceil \frac{3\phi}{\pi} \right\rceil \frac{\pi}{3} - \frac{\pi}{6} - \phi\right)}$, where $\Gamma_{\mathbf{x}_i} = \{7, m\}$, and m depends on ϕ.

3) Interference from Users Located Within Reference Cell: We now have $d_0(\mathbf{x}_i) = r$. For $-\frac{\pi}{6} \le \phi < \frac{\pi}{6}$, $\Gamma_{\mathbf{x}_i} = \{0, 1\}$, and $d_1(\mathbf{x}_i) = \sqrt{D^2 + r^2 - 2rD\cos(\phi)}$. For $\frac{\pi}{6} \le \phi < \frac{\pi}{2}$, $\Gamma_{\mathbf{x}_i} = \{0, 2\}$ and $d_2(\mathbf{x}_i) = \sqrt{D^2 + r^2 - 2rD\cos\left(\phi - \frac{\pi}{3}\right)}$. For $\frac{\pi}{2} \le \phi < \frac{5\pi}{6}$, we have $\Gamma_{\mathbf{x}_i} = \{0, 3\}$ and $d_3(\mathbf{x}_i) = \sqrt{D^2 + r^2 - 2rD\cos\left(\phi - \frac{2\pi}{3}\right)}$. For $\frac{5\pi}{6} \le \phi < \frac{7\pi}{6}$, we have $\Gamma_{\mathbf{x}_i} = \{0, 4\}$ and $d_4(\mathbf{x}_i) = \sqrt{D^2 + r^2 - 2rD\cos(\phi - \pi)}$. For $\frac{7\pi}{6} \le \phi < \frac{3\pi}{2}$, we have $\Gamma_{\mathbf{x}_i} = \{0, 5\}$ and $d_5(\mathbf{x}_i) = \sqrt{D^2 + r^2 - 2rD\cos\left(\phi - \frac{4\pi}{3}\right)}$. For $\frac{3\pi}{2} \le \phi < \frac{11\pi}{6}$, we have $\Gamma_{\mathbf{x}_i} = \{0, 6\}$ and $d_6(\mathbf{x}_i) = \sqrt{D^2 + r^2 - 2rD\cos\left(\phi - \frac{5\pi}{3}\right)}$.

D. Cell Selection With $|\Gamma_{\mathbf{x}_i}| = 3$ Cells

1) Interference from First-Tier Cell 1: As before, $\Gamma_{\mathbf{x}_i}$ depends on ϕ. For $0 \le \phi < \frac{\pi}{3}$, $\Gamma_{\mathbf{x}_i} = \{0, 1, 2\}$; for $\frac{\pi}{3} \le \phi < \frac{2\pi}{3}$, $\Gamma_{\mathbf{x}_i} = \{1, 2, 8\}$; for $\frac{2\pi}{3} \le \phi < \pi$, $\Gamma_{\mathbf{x}_i} = \{1, 8, 7\}$; and so on. The distances to the various cells in $\Gamma_{\mathbf{x}_i}$ can be calculated accordingly. Note that for $-\frac{\pi}{3} \le \phi < \frac{\pi}{3}$, the reference cell is itself in the handoff set.

2) Interference from Second-Tier Cell 7: As before, the reference cell is never in the handoff set. The handoff sets and the corresponding distances from the cell centers can again be enumerated as a function of ϕ. Clearly, any cell included in $\Gamma_{\mathbf{x}_i}$ when $|\Gamma_{\mathbf{x}_i}| = 2$ will also be included in $\Gamma_{\mathbf{x}_i}$ when $|\Gamma_{\mathbf{x}_i}| = 3$. The details are skipped to conserve space.

Corresponding equations for $|\Gamma_{\mathbf{x}_i}| \ge 4$ can also be written. However, they get more involved are not shown here.

IV. Simulations

We now compare the accuracy achieved in modeling the uplink inter-cell interference. The moment-matched lognormal approximation, which uses circular cell shapes for analytical tractability, is compared with Monte Carlo simulations that use hexagonal cell shapes and generate 3×10^5 sample points. In each sample point, the user locations and the number of users per cell are generated as per the spatial Poisson process described in Sec. II-A for hexagonal cell shapes. The shadowing from different users to different BSs are generated as independent lognormal random variables. Each user first chooses its serving BS and then sets its transmit power to compensate for shadowing and path loss. Also plotted are the Gaussian approximation and the Edgeworth approximation, which uses the third cumulant.

Our comparison is based on the cumulative distribution function (CDF) and the complementary CDF (CCDF), as is typically done in the literature [3], [5], [10]. Small values of the CDF reveal the accuracy in tracking the head portion (small interference values) of the probability distribution. Similarly, small values of the CCDF reveal the accuracy in tracking the tail portion (large interference values) of the interference probability distribution. The system parameters used in the simulations are: path loss exponent $\eta = 4$, power control threshold $\gamma = 8$ dB, lognormal dB standard deviation $\sigma = 6$, cell radius $R = 400$ m, and $D = 800$ m.

A large number of combinations are possible for the figures that depend on the choice of handoff set size, whether first- or second-tier interference or both are being considered, and the average number of users per cell. Due to space constraints, we illustrate a subset of these combinations to bring out the salient points about the approach developed in this paper.

The CDF and CCDF of the inter-cell interference power from a first-tier cell with power control only are plotted in Figure 2. It can be seen that the lognormal is a better approximation for the CDF as well as the CCDF than the Gaussian and Edgeworth approximations. While not perfect, the lognormal is two orders of magnitude more accurate in approximating the CDF than the Gaussian. This is because the Gaussian CDF, say with mean μ_G and variance σ_G^2, in fact, saturates for small interference values at $Q(\mu_G/\sigma_G)$. On the other hand, this saturation does not happen for the lognormal CDF. As the interference, R, tends to 0, the lognormal CDF equals $Q\left(\frac{|\log_e(R)|}{\sigma_{\text{eq}}}\right)$, which also tends to 0. The lognormal approximation is also much better in tracking the CCDF of the interference.

The case with power control and cell selection is considered in Figure 3, which plots the CDF and CCDF of the inter-cell interference from a first-tier cell for $K = \lambda A = 30$. While the Edgeworth approximation is better than the Gaussian approximation, the lognormal continues to better model the simulated CDF and the CCDF. Interestingly, the lognormal's accuracy in tracking the CDF improves as K increases from

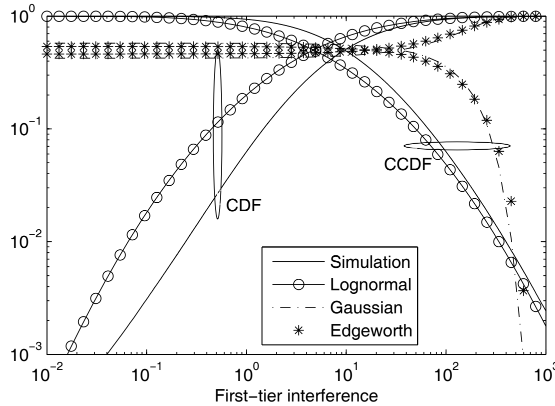

Fig. 2. CDF and CCDF of interference power from a first-tier cell with power control and 10 users per cell on average.

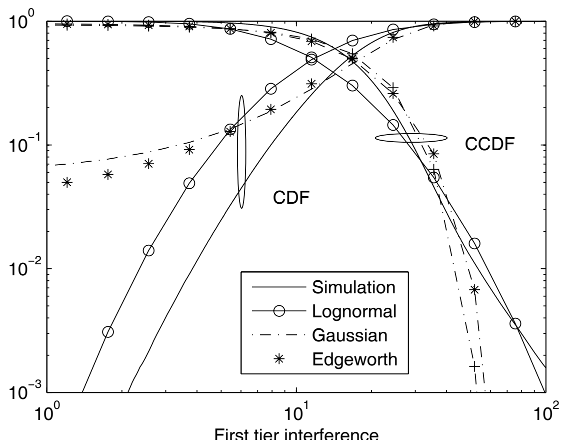

Fig. 3. CDF and CCDF of interference power from a first-tier cell with power control and cell selection with $|\Gamma_{\mathbf{x}_i}| = 3$ and 30 users per cell on average.

10 to 30, and starts degrading only when K exceeds 50 (figure not shown).

Finally, the statistics of the *total interference from all first-tier and second-tier cells* in the presence of only power control and both power control and cell selection (with two cells in the handoff set) is plotted in Figures 4 and 5, respectively. The interference also includes contributions, if any, from users located within the reference cell. Now, the difference between the Edgeworth and Gaussian approximations diminishes. While the lognormal remains a better model than the Gaussian, the former's relative inaccuracy increases as the total number of interferers has increased considerably. Thus, several observations made in the literature when the number of lognormal summands (and their parameters) is fixed [3], [5], [10] carry over to our problem despite the additional sources of randomness in the uplink.

V. Conclusions

We developed an alternate characterization of the statistical distribution of the inter-cell interference power observed in the uplink of CDMA systems. For this, we extended the moment-matching approach to determine the parameters of the approximating lognormal distribution. The approach captured

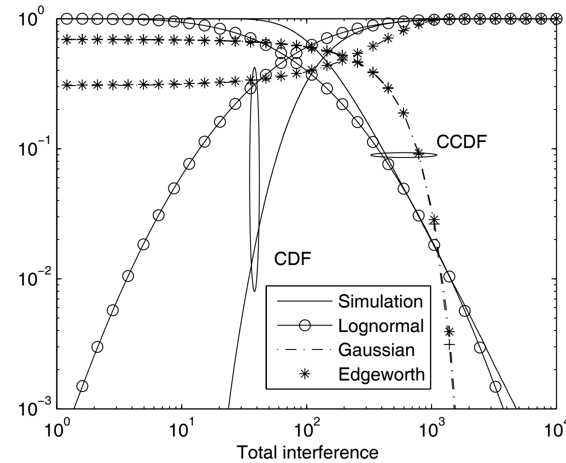

Fig. 4. CDF and CCDF of total interference power from all first-tier and second-tier cells with power control and 10 users per cell on average.

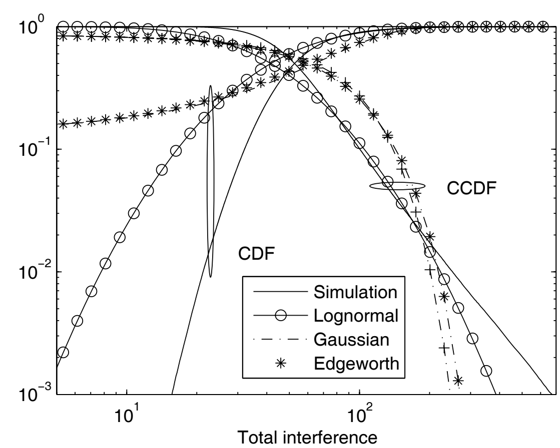

Fig. 5. CDF and CCDF of the total inter-cell interference power (from first-tier and second-tier cells and from within reference cell) for $|\Gamma_{\mathbf{x}_i}| = 2$ and 10 users per cell on average.

the underlying wireless channel propagation parameters, and also accounted for the additional randomness introduced in the shadowed interference power due to randomness in the number of users and their spatial locations. Both power control and cell selection were accommodated in the approach. The lognormal, while not perfect, turned out to be considerably more accurate than the Gaussian in approximating both the CDF and CCDF of the interference power. Interestingly, several insights for the well-studied case where the number of lognormal summands is fixed (and so are their parameters) carry over to the uplink scenario as well. The results also show that scope for further improvement in the accuracy of the lognormal approximation remains.

References

[1] C. C. Chan and S. V. Hanly, "Calculating the outage probability in a CDMA network with spatial Poisson traffic," *IEEE Trans. Veh. Technol.*, vol. 50, pp. 183-204, Jan. 2001.

[2] D. Tse and P. Vishwanath, *Fundamentals of Wireless Communications*. Cambridge University Press, 2005.

[3] N. B. Mehta, J. Wu, A. F. Molisch, and J. Zhang, "Approximating a sum of random variables with a lognormal," *IEEE Trans. Wireless Commun.*, vol. 6, pp. 2690-2699, July 2007.

[4] A. F. Molisch, *Wireless Communications*. Wiley-IEEE Press, 2005.

[5] N. C. Beaulieu, A. Abu-Dayya, and P. McLance, “Estimating the distribution of a sum of independent lognormal random variables," *IEEE Trans. Commun.*, vol. 43, pp. 2869-2873, 1995.

[6] W. A. Janos, “Tail of the distributions of sums of lognormal variates," *IEEE Trans. Inf. Theory*, vol. IT-16, pp. 299-302, 1970.

[7] L. F. Fenton, “The sum of lognormal probability distributions in scatter transmission systems," *IRE Trans. Commun. Syst.*, vol. CS-8, pp. 57-67, 1960.

[8] S. Schwartz and Y. Yeh, “On the distribution function and moments of power sums with lognormal components," *Bell Syst. Tech. J.*, vol. 61, pp. 1441-1462, 1982.

[9] D. C. Schleher, “Generalized Gram-Charlier series with application to the sum of log-normal variates," *IEEE Trans. Inf. Theory*, pp. 275-280, 1977.

[10] N. C. Beaulieu and Q. Xie, “An optimal lognormal approximation to lognormal sum distributions," *IEEE Trans. Veh. Technol.*, vol. 53, pp. 479-489, 2004.

[11] J. F. C. Kingman, *Poisson Processes*. Oxford University Press, 1993.

[12] J. Zhang and J. G. Andrews, “Distributed antenna systems with randomness," *IEEE Trans. Wireless Commun.*, vol. 7, pp. 3636-3646, Sept. 2008.

[13] R. S. Karlsson and D. Everitt, “Teletraffic capacity of hierarchical CDMA cellular networks," in *Proc. Multiaccess, Mobility Teletraffic Wireless Commun.*, pp. 341-352, 2000.

[14] G. L. Stüber, *Principles of Mobile Communications*. Kluwer Academic Publishers, 1996.

[15] M. Abramowitz and I. Stegun, *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*. Dover, 9th ed., 1972.