Generalizing the Amplify-and-Forward Relay Gain Model: An Optimal SEP Perspective

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***Abstract*—Two models for AF relaying, namely, fixed gain and fixed power relaying, have been extensively studied in the literature given their ability to harness spatial diversity. In fixed gain relaying, the relay gain is fixed but its transmit power varies as a function of the source-relay channel gain. In fixed power relaying, the relay transmit power is fixed, but its gain varies. We revisit and generalize the fundamental two-hop AF relaying model. We present an optimal scheme in which an average power constrained AF relay adapts its gain and transmit power to minimize the symbol error probability (SEP) at the destination. Also derived are insightful and practically amenable closed-form bounds for the optimal relay gain. We then analyze the SEP of MPSK, derive tight bounds for it, and characterize the diversity order for Rayleigh fading. Also derived is an SEP approximation that is accurate to within 0.1 dB. Extensive results show that the scheme yields significant energy savings of 2.0-7.7 dB at the source and relay. Optimal relay placement for the proposed scheme is also characterized, and is different from fixed gain or power relaying. Generalizations to MQAM and other fading distributions are also discussed.**

***Index Terms*—Cooperative systems, spatial diversity, relays, MPSK, MQAM, power constraint, modulation, fading channels, amplify-and-forward, symbol error probability, relay placement.**

I. Introduction

RELAY-BASED cooperative communication exploits spatial diversity to combat wireless fading, and is considered to be one of the key technologies for next generation wireless systems [1]. Using transmitters and receivers that have only one antenna, relaying provides multiple independent fading paths for the signal transmitted by a source to reach its destination [2]. Among the many relaying schemes that have been studied in the literature, amplify-and-forward (AF) relaying is a classical scheme, and is popular because the relay does not have to decode the source's message [3]–[13]. It is a two-phase cooperation protocol. The source transmits in the first phase, and in the second phase, the AF relay amplifies the noisy and faded signal it receives from the source and transmits it to the destination.

Two AF relaying policies have been extensively investigated in the literature, namely, fixed power relaying [3], [5], [6], [8], [9], [13], [14] and fixed gain relaying [10], [12], [15]–[17]. In fixed power relaying, the relay adjusts its gain as a function of the source-relay (SR) channel gain such that its instantaneous transmit power, when averaged over the noise added by the relay and the data symbols, is fixed. On the other hand, in fixed gain relaying, the relay gain is fixed. As a result, its instantaneous transmit power depends on the SR channel gain. Notice that in the above two policies, the relay adapts its gain or power as a function of its SR channel gain. While these policies are intuitive, they do not optimize an end objective such as the symbol error probability (SEP) at the destination.

In this paper, we reexamine the fundamental two-hop building block of an AF relaying system that consists of a source, a destination, and a relay. We generalize the AF relaying model by considering a relay that adapts both its relay gain and transmit power as a function of its SR and relay-destination (RD) channel gains. However, the relay remains non-regenerative and does not decode the signal it receives from the source. We derive the optimal gain and power adaptation rule at the average power constrained relay that minimize the SEP of MPSK at the destination receiver. We shall refer to this as adaptive relay gain and transmit power (ARGTP) relaying. Thereafter, we discuss generalizations to MQAM and to different channel fading models.

We then analyze the SEP of the proposed AF relaying policy. We derive two upper bounds for the SEP, which trade-off evaluation complexity with tightness. We also derive an accurate approximation for the SEP, which is less than 0.1 dB away from the exact value. Extensive simulation results are presented to benchmark the proposed scheme's performance with conventional AF relaying schemes. We find that ARGTP relaying, which optimally exploits local knowledge of the SR and RD channel gains, significantly outperforms both fixed power and fixed gain relaying. For example, for QPSK, with unit average channel power gains and equal source and relay powers, ARGTP relaying delivers energy savings of 2.0 dB to 2.4 dB over fixed power relaying and fixed gain relaying at an SEP of 10^{-2}. The corresponding energy savings increase to 4.3 dB to 7.7 dB when the direct source-destination (SD) link is absent. The results also show that the finite-SNR diversity order [18], [19], which quantifies how the system harnesses the spatial diversity at finite SNRs, of ARGTP relaying is better than the conventional AF relaying schemes over a wide range of SNRs.

While minimizing the SEP or maximizing throughput have also been investigated in [16], [20], [21], the focus was on optimally allocating power between the source and relay; either fixed power or fixed gain relaying was assumed. A

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different generalization was considered in [22], in which the relay gain was varied a function of the SR and RD channel gains to maximize the average SNR at the destination. However, our design objectives and the depth of our results are different for the following reasons:

- We focus on minimizing the fading-averaged SEP, while [22] focuses on maximizing the average SNR at the destination. The SEP is an end-to-end performance metric while the SNR is an intermediate metric. While a high instantaneous SNR implies a low instantaneous SEP, a high fading-averaged SNR need not imply a low fading-averaged SEP because the SEP is a non-linear function of the SNR.
- While [22] shows that quasi-convexity based techniques can be used to numerically determine the relay gain as a function of the channel gains, we instead analytically characterize the SEP-optimal relay gain policy and show that it is functionally quite different from fixed power or fixed gain relaying.
- We also develop two upper and lower bounds for the relay gain that are theoretically insightful and make it feasible to practically implement ARGTP relaying. Such an approach has not been pursued in [22].
- For ARGTP relaying, we analyze the SEP, derive analytically simpler upper bounds and a tight approximation for the SEP, and characterize the diversity order for Rayleigh fading. Such an SEP analysis was not attempted in [22].

The paper is organized as follows. Section II describes our system model. Section III derives the optimal ARGTP relaying rule, whose SEP is analyzed in Section IV. Our results and conclusions follow in Sections V and VI, respectively. Mathematical details are relegated to the Appendix.

II. System Model

Figure 1 depicts a system consisting of a source S, a destination D, and a half-duplex relay R. Each node is equipped with a single transmit or receive antenna. The SR, RD, and SD channels are assumed to be block fading channels that undergo independent frequency-flat Rayleigh fading. However, they need not be statistically identical. All transmissions occur over the same bandwidth. The relay is assumed to know its local SR and RD channel gains [22]–[24]. In practice, these can be acquired by a training protocol [11], [25], [26]. However, the relay need not know the SD channel gain.

We use the following notation henceforth. The probability of an event A is denoted by $\Pr(A)$. For a random variable (RV) X, its probability density function (PDF), expectation, and variance are denoted by $p_X(x)$, $\mathbf{E}[X]$, and $\mathbf{var}[X]$, respectively. $\mathcal{CN}(0,\sigma^2)$ represents a zero-mean circular symmetric complex Gaussian RV with variance σ^2. The complex conjugate of x is denoted by x^*.

A. Cooperative AF Protocol

The cooperative AF relaying protocol occurs over two phases. In the first phase, the source broadcasts a data symbol α that is drawn with equal probability from the M-ary PSK

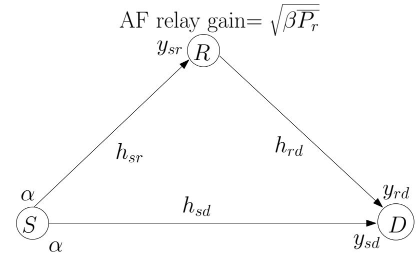

Fig. 1. Average relay transmit power constrained and variable gain AF relay model

(MPSK) constellation of size M. The received signals y_{sd} and y_{sr} at the destination and relay, respectively, are given by

$$y_{sd} = \sqrt{P_s} h_{sd}\alpha + n_{sd}, \tag{1}$$
$$y_{sr} = \sqrt{P_s} h_{sr}\alpha + n_{sr}, \tag{2}$$

where P_s is the source transmit power, h_{sd} is the SD channel gain and is a $\mathcal{CN}(0,\sigma_{sd}^2)$ RV, and h_{sr} is the SR channel gain and is a $\mathcal{CN}(0,\sigma_{sr}^2)$ RV. Without loss of generality (w.l.o.g.), $|\alpha|^2 = 1$. Further, w.l.o.g., the additive noise terms n_{sr} and n_{sd} are modeled as $\mathcal{CN}(0,1)$ RVs, and are independent of each other and the channel gains.

In the second phase, the relay amplifies the signal it receives, y_{sr}, by a factor $\sqrt{\beta \overline{P}_r}$, where $\overline{P}_r$ is the average relay transmit power. Therefore, the destination receives the signal $y_{rd} = \sqrt{\beta \overline{P}_r} h_{rd} y_{sr} + n_{rd}$, which can be shown to be equal to

$$y_{rd} = \sqrt{\beta \overline{P}_r P_s} h_{sr} h_{rd}\alpha + \sqrt{\beta \overline{P}_r} h_{rd} n_{sr} + n_{rd}. \tag{3}$$

Let $\gamma_{sd} \triangleq |h_{sd}|^2 P_s$, $\gamma_{sr} \triangleq |h_{sr}|^2 P_s$, and $\gamma_{rd} \triangleq \overline{P}_r |h_{rd}|^2$. Further, $\overline{\gamma}_{sd} \triangleq \mathbf{E}[\gamma_{sd}]$, $\overline{\gamma}_{sr} \triangleq \mathbf{E}[\gamma_{sr}]$, and $\overline{\gamma}_{rd} \triangleq \mathbf{E}[\gamma_{rd}]$.

B. General AF Relaying Model

The main difference in our model lies in the AF relay gain. In fixed power AF relaying [3], [5], [9], the relay gain for the above relay transmission model is given by $\beta = \frac{1}{\gamma_{sr}+1}$. On the other hand, in fixed gain AF relaying [4], [11], the relay gain equals $\beta = \frac{1}{\overline{\gamma}_{sr}+1}$.

In our model, however, the relay gain is a function of the local SR and RD channel gains, γ_{sr} and γ_{rd}. As mentioned, we do not assume that the relay knows the SD channel gain, as this would require channel state feedback from the destination. Hence, the relay gain will be denoted by $\beta(\gamma_{sr},\gamma_{rd})$. The relay is subject to an average relay transmit power constraint, which is $\mathbf{E}[\beta(\gamma_{sr},\gamma_{rd})(\gamma_{sr}+1)] = 1$. The destination coherently determines α using its two observables y_{sd} and y_{rd}. We assume that it knows the SR, RD, and SD channel gains [3], [5], [6]. The SNR γ_E at the destination receiver when it employs maximal ratio combining is given by [6], [20] $\gamma_E = \gamma_{sd} + \frac{\gamma_{sr}\gamma_{rd}}{\gamma_{rd}+\beta(\gamma_{sr},\gamma_{rd})^{-1}}$.

III. Optimal ARGTP Relaying

We first derive the optimum relay gain function β_{opt}. The fading-averaged SEP for MPSK at the destination is given by [27, (8.23)]

$$\text{SEP} = \frac{1}{\pi}\int_0^{\left(\frac{M-1}{M}\right)\pi} \mathbf{E}\left[\exp\left(-\gamma_E \frac{m}{\sin^2\theta}\right)\right] d\theta, \tag{4}$$

where $m = \sin^2\left(\frac{\pi}{M}\right)$. Averaging over γ_{sd}, which is an exponential RV that is independent of γ_{sr}, γ_{rd}, and β, yields

$$\mathrm{SEP} = \frac{1}{\pi}\int_0^{\left(\frac{M-1}{M}\right)\pi} \frac{\mathbf{E}\left[\exp\left(-\frac{\gamma_{sr}\gamma_{rd}}{\gamma_{rd}+\beta(\gamma_{sr},\gamma_{rd})^{-1}}\frac{m}{\sin^2\theta}\right)\right]}{1+\overline{\gamma}_{sd}\frac{m}{\sin^2\theta}}\, d\theta. \quad (5)$$

The SEP expression in (5) cannot be simplified further because the relay gain is now a function of γ_{sr} and γ_{rd}, and it is this function that we seek to optimize. To gain further insights, we derive below an analytically tractable upper bound for the SEP and shall minimize it instead. Using the inequality $\sin^2\theta \le 1$ only for the term inside the expectation in the integrand in (5), we get

$$\mathrm{SEP} \le \mathrm{SEP}_0 \mathbf{E}\left[\exp\left(-\frac{m\gamma_{sr}\gamma_{rd}}{\gamma_{rd}+\beta(\gamma_{sr},\gamma_{rd})^{-1}}\right)\right], \quad (6)$$

where SEP_0 captures the contribution of the direct SD link and is given by

$$\mathrm{SEP}_0 = \frac{1}{\pi}\int_0^{\left(\frac{M-1}{M}\right)\pi}\left(1+\overline{\gamma}_{sd}\frac{m}{\sin^2\theta}\right)^{-1} d\theta. \quad (7)$$

Using the variable substitution $t = \cot(\theta)$ and simplifying, it can be shown that SEP_0 reduces to the following expression:

$$\mathrm{SEP}_0 = \frac{M-1}{M} - \frac{1}{\sqrt{1+\frac{m^{-1}}{\overline{\gamma}_{sd}}}}\left(\frac{1}{2}+\frac{1}{\pi}\arctan\sqrt{\frac{1-m}{m+\frac{1}{\overline{\gamma}_{sd}}}}\right). \quad (8)$$

A. AF Relaying Optimization

Since SEP_0 does not depend on the relay gain, the SEP minimization problem now reduces to finding the optimal function $\beta_{\text{opt}} : (\mathbb{R}^+)^2 \to \mathbb{R}$, which is a function of two variables γ_{sr} and γ_{rd}, that minimizes the expectation term in (6). Mathematically, the optimization problem can be stated as

$$\min_{\beta} \quad \mathbf{E}\left[\exp\left(-\frac{m\gamma_{sr}\gamma_{rd}}{\gamma_{rd}+\beta(\gamma_{sr},\gamma_{rd})^{-1}}\right)\right] \quad (9)$$

$$\text{s. t.} \quad \mathbf{E}\left[\beta(\gamma_{sr},\gamma_{rd})\,(\gamma_{sr}+1)\right] = 1, \text{ and} \quad (10)$$

$$\beta(\gamma_{sr},\gamma_{rd}) \ge 0, \forall\ \gamma_{sr} \ge 0, \gamma_{rd} \ge 0. \quad (11)$$

Notice that fixed power and fixed gain relaying are feasible solutions of this problem. The optimal solution is as follows.

Result *1:* Let

$$\phi(x) \triangleq \exp\left(\frac{m\gamma_{sr}\gamma_{rd}}{\gamma_{rd}+x^{-1}}\right)(x\gamma_{rd}+1)^2 - \frac{m\gamma_{sr}\gamma_{rd}}{\lambda(\gamma_{sr}+1)}. \quad (12)$$

For $\gamma_{rd} \ge \mathcal{B}(\gamma_{sr})$, $\phi(x)$ has a unique positive root $x_0(\gamma_{sr},\gamma_{rd})$, and

$$\beta_{\text{opt}}(\gamma_{sr},\gamma_{rd}) = \begin{cases} x_0(\gamma_{sr},\gamma_{rd}), & \gamma_{rd} \ge \mathcal{B}(\gamma_{sr}) \\ 0, & \text{otherwise} \end{cases}, \quad (13)$$

where

$$\mathcal{B}(\gamma_{sr}) \triangleq \frac{\lambda}{m}\left(1+\frac{1}{\gamma_{sr}}\right). \quad (14)$$

The Lagrange multiplier λ is chosen to satisfy the average power constraint in (10).

Proof: The proof is relegated to Appendix A. ■

B. Closed-form Insights into Optimum AF Relay Gain

One problem with the transcendental equation in (12) is that x appears in it inside the exponential term as well as in the term outside the exponential. Therefore, it needs to be computed numerically by solving (12) for each realization of γ_{sr} and γ_{rd}. We now present a lower bound $\beta_l(\gamma_{sr},\gamma_{rd})$ and an upper bound $\beta_u(\gamma_{sr},\gamma_{rd})$ for $\beta_{\text{opt}}(\gamma_{sr},\gamma_{rd})$. These provide new insights into the structure of the optimal AF relaying scheme and have practical utility as described below.

Result *2:* The optimal AF relay gain β_{opt} is lower bounded as follows:

$$\beta_{\text{opt}}(\gamma_{sr},\gamma_{rd}) \ge \beta_l(\gamma_{sr},\gamma_{rd}) = \begin{cases} \frac{\sqrt{\frac{m\gamma_{sr}\gamma_{rd}}{\lambda(\gamma_{sr}+1)}\exp\left(-\frac{m\gamma_{sr}}{2}\right)}-1}{\gamma_{rd}}, & \gamma_{rd} \ge \mathcal{B}_1(\gamma_{sr}) \\ 0, & \text{otherwise} \end{cases}, \quad (15)$$

where

$$\mathcal{B}_1(\gamma_{sr}) = \mathcal{B}(\gamma_{sr})\exp\left(m\gamma_{sr}\right). \quad (16)$$

Proof: The proof is given in Appendix B. ■

Result *3:* The optimal AF relay gain β_{opt} is upper bounded as follows:

$$\beta_{\text{opt}}(\gamma_{sr},\gamma_{rd}) \le \beta_u(\gamma_{sr},\gamma_{rd}) = \begin{cases} \frac{-1+\sqrt{1+\left[\frac{m\gamma_{sr}\gamma_{rd}}{\lambda(\gamma_{sr}+1)}-1\right](1+m^2\gamma_{sr}^2)}}{\gamma_{rd}(1+m^2\gamma_{sr}^2)}, & \gamma_{rd} \ge \mathcal{B}(\gamma_{sr}) \\ 0, & \text{otherwise} \end{cases}. \quad (17)$$

Proof: The proof is given in Appendix C. ■

Figure 2 plots β_{opt}, β_u, and β_l as a function of γ_{rd} for $\gamma_{sr} = 0.5$ dB for QPSK ($M = 4$). It verifies that β_u is an upper bound and β_l is a lower bound. Notice that while the bounds are not tight over the entire range of γ_{rd}, their variation is qualitatively similar to β_{opt}. We see that the optimal policy shuts off the relay for small values of γ_{rd}. This makes intuitive sense because the relay would otherwise need to transmit at a high power and expend a significant amount of its energy in order to overcome a weak RD channel and make its signal heard by the destination. The optimal policy also reduces the relay transmit power for large values of γ_{rd}. As can be seen from (15) and (17), both β_l and β_u, and hence β_{opt}, decrease as $1/\sqrt{\gamma_{rd}}$ for large γ_{rd}. This also makes intuitive sense because there is limited benefit in making the relay transmit at a high transmit power when the RD channel itself boosts the signal sent by the relay. This behavior is different from fixed power relaying, in which the relay's transmit power does not depend on the SR and RD channel gains.

C. Implementability of ARGTP Relaying and Practical Utility of Bounds

Using β_{opt} is difficult from an implementation perspective in an AF relay, since it would require the use of an unwieldy two-dimensional look-up table to determine $\beta_{\text{opt}}(\gamma_{sr},\gamma_{rd})$ as a function of γ_{sr} and γ_{rd}. Given their qualitatively similar behavior as β_{opt}, the relay can instead use as its relay gain β_l or β_u, whose dependence on the SR and RD channel gains is available in closed-form in (15) and (17). While these relay gain expressions are more involved than those for fixed

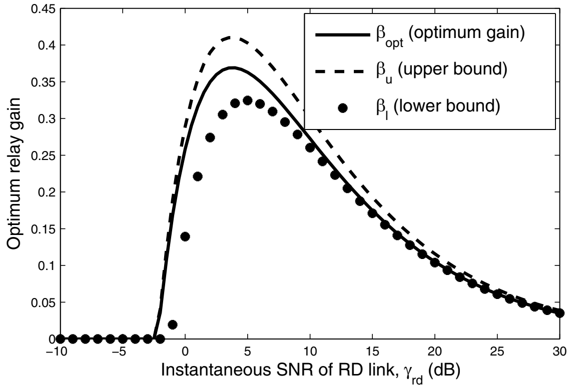

Fig. 2. Comparison of optimum relay gain, β_{opt}, its upper bound, β_u, and its lower bound, β_l, as a function of the instantaneous RD link SNR, γ_{rd} ($\gamma_{sr} = 0.5$ dB, $\lambda = 0.1$, and QPSK).

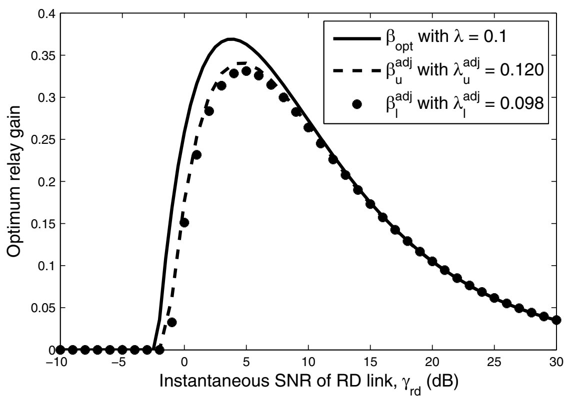

Fig. 3. Comparison of β_{opt}, β_u^{adj}, and β_l^{adj} as a function of the instantaneous RD link SNR, γ_{rd} ($\gamma_{sr} = 0.5$ dB, $\lambda = 0.1$, and QPSK).

power and fixed gain relaying, they are in terms of elementary functions and are easily computable.

In order to meet the average power constraint, the value of the Lagrange multiplier needs to be chosen depending on whether β_u or β_l is used. If λ is the value of the Lagrange multiplier at which the power constraint is satisfied by β_{opt}, then $\beta_{\text{opt}} \leq \beta_u$ implies that $\mathbf{E}\left[\beta_u(\gamma_{sr},\gamma_{rd})(\gamma_{sr}+1)\right] \geq 1$ and $\beta_{\text{opt}} \geq \beta_l$ implies that $\mathbf{E}\left[\beta_l(\gamma_{sr},\gamma_{rd})(\gamma_{sr}+1)\right] \leq 1$. Therefore, let λ_u^{adj} denote the value of the Lagrange multiplier such that the relay that uses β_u meets the average power constraint. Similarly, let λ_l^{adj} denote the value of the Lagrange multiplier such that the relay that uses β_l meets the average power constraint. Figure 3 plots β_{opt}, β_l^{adj}, and β_u^{adj} as a function of γ_{rd}. We observe that β_u^{adj} and β_l^{adj} now track β_{opt} better.

Note that the Lagrange multiplier depends on the channel fading statistics of the SR and RD links and not their instantaneous channel gains. It, therefore, needs to be computed only once. Its occurrence in the optimal relay gain expressions is typical of optimal solutions to problems in which a node is subject to an average transmit power constraint, e.g., [28].

D. Optimal Relaying for MQAM and Other Fading Distributions

The optimal relay gain policy for MQAM is of the same form as for MPSK for the following reason. The fading-averaged SEP for MQAM at the destination can be shown to be upper bounded by [27, (8.12)]

$$\begin{aligned}\text{SEP} &\leq \frac{M-1}{M}\mathbf{E}\left[\exp\left(-\frac{3}{2(M-1)}\gamma_E\right)\right],\\ &= \frac{M-1}{M\left(1+\frac{3\overline{\gamma}_{sd}}{2(M-1)}\right)}\mathbf{E}\left[\exp\left(-\frac{\frac{3}{2(M-1)}\gamma_{sr}\gamma_{rd}}{\gamma_{rd}+\beta(\gamma_{sr},\gamma_{rd})^{-1}}\right)\right].\end{aligned}$$

This has the same form as (6) except for a different scaling constant in front of the expectation, which does not affect the optimization problem in (9), and with m replaced by $\frac{3}{2(M-1)}$. Therefore, for MQAM, the expressions for the optimal relay gain β_{opt} and its bounds β_l and β_u are the same as those in (13), (15), and (17), respectively, except that m is replaced by $\frac{3}{2(M-1)}$. The value of λ is chosen to meet the average power constraint.[1]

The results in (13), (17), and (15), in fact, hold for other channel fading distributions as well. This can be seen from the proof of Result 1 in Appendix A. In it, the expectation operator, which averages over the different SR and RD channel gains, gets dropped in the unconstrained optimization problem in (38). Consequently, $\beta(\gamma_{sr},\gamma_{rd})$ is optimized for each γ_{sr} and γ_{rd}. The fading distribution only affects the value of λ, which is chosen to meet the average power constraint.

IV. SEP Analysis of Optimal ARGTP Relaying

We now analyze the SEP of ARGTP relaying. First, analytically tractable upper bounds, which are based on β_u and trade off evaluation complexity with tightness, are derived. Thereafter, an accurate SEP approximation is derived using an alternate approach based on β_l. The diversity order is then characterized for Rayleigh fading.

Expanding the expectation in the integrand in (5) results in the following exact SEP expression, which is in the form of a triple-integral over θ, γ_{sr}, and γ_{rd}:

$$\begin{aligned}\text{SEP} &= \frac{1}{\pi\overline{\gamma}_{sr}\overline{\gamma}_{rd}}\\ &\times\int_0^\infty\int_0^\infty\int_0^{\left(\frac{M-1}{M}\right)\pi}\frac{\exp\left(-\frac{\gamma_{sr}\gamma_{rd}}{\gamma_{rd}+\beta_{\text{opt}}(\gamma_{sr},\gamma_{rd})^{-1}}\frac{m}{\sin^2\theta}\right)}{1+\overline{\gamma}_{sd}\frac{m}{\sin^2\theta}}\\ &\quad\times\exp\left(-\frac{\gamma_{sr}}{\overline{\gamma}_{sr}}\right)\exp\left(-\frac{\gamma_{rd}}{\overline{\gamma}_{rd}}\right)\,d\theta\,d\gamma_{rd}\,d\gamma_{sr}.\end{aligned}\qquad(18)$$

Further, using the inequality $\sin^2\theta \leq 1$, we get the following upper bound SEP_u:

$$\begin{aligned}\text{SEP}_u = \frac{\text{SEP}_0}{\overline{\gamma}_{sr}\overline{\gamma}_{rd}}\int_0^\infty\int_0^\infty &\exp\left(-\frac{m\gamma_{sr}\gamma_{rd}}{\gamma_{rd}+\beta_{\text{opt}}(\gamma_{sr},\gamma_{rd})^{-1}}\right)\\ &\times\exp\left(-\frac{\gamma_{sr}}{\overline{\gamma}_{sr}}\right)\exp\left(-\frac{\gamma_{rd}}{\overline{\gamma}_{rd}}\right)\,d\gamma_{sr}\,d\gamma_{rd}.\end{aligned}\qquad(19)$$

[1]In general, for several constellations, the SEP is well approximated by [28] $a_1\exp(-a_2\gamma_E)$. Thus, β_{opt} for these constellations also has the same form as for MPSK except that m is replaced by a_2.

It is still in the form of a double-integral. We, therefore, simplify it further below.

From (12), we know that $\exp\left(\frac{m\gamma_{sr}\gamma_{rd}}{\gamma_{rd}+\beta_{\text{opt}}(\gamma_{sr},\gamma_{rd})^{-1}}\right) = \frac{m\gamma_{sr}\gamma_{rd}}{\lambda(\gamma_{sr}+1)}\left(\beta_{\text{opt}}(\gamma_{sr},\gamma_{rd})\gamma_{rd}+1\right)^{-2}$, for $\gamma_{rd} \geq \mathcal{B}(\gamma_{sr})$, and is equal to 1, otherwise. Therefore, the expression for SEP_u above becomes

$$\begin{aligned}\text{SEP}_u &= \frac{\text{SEP}_0}{\overline{\gamma}_{sr}\overline{\gamma}_{rd}} \\ &\times \int_0^\infty \int_0^{\mathcal{B}(\gamma_{sr})} \exp\left(-\frac{\gamma_{sr}}{\overline{\gamma}_{sr}}\right)\exp\left(-\frac{\gamma_{rd}}{\overline{\gamma}_{rd}}\right) d\gamma_{sr}\, d\gamma_{rd} \\ &+ \frac{\text{SEP}_0}{\overline{\gamma}_{sr}\overline{\gamma}_{rd}} \int_0^\infty \int_{\mathcal{B}(\gamma_{sr})}^\infty \frac{\lambda(\gamma_{sr}+1)}{m\gamma_{sr}\gamma_{rd}} (\beta_{\text{opt}}(\gamma_{sr},\gamma_{rd})\gamma_{rd}+1)^2 \\ &\times \exp\left(-\frac{\gamma_{sr}}{\overline{\gamma}_{sr}}\right)\exp\left(-\frac{\gamma_{rd}}{\overline{\gamma}_{rd}}\right) d\gamma_{sr}\, d\gamma_{rd}. \quad (20)\end{aligned}$$

Let the first term in the expression for SEP_u above be denoted by I_1. Using [29, (3.324.1)], it can be shown that I_1 simplifies in closed-form to

$$\begin{aligned}I_1 = \text{SEP}_0 \Bigg[1 - \frac{2}{\overline{\gamma}_{sr}}\exp\left(-\frac{\lambda}{m\overline{\gamma}_{rd}}\right)\sqrt{\frac{\lambda\overline{\gamma}_{sr}}{m\overline{\gamma}_{rd}}} \\ \times K_1\left(\sqrt{\frac{4\lambda}{m\overline{\gamma}_{sr}\overline{\gamma}_{rd}}}\right)\Bigg], \quad (21)\end{aligned}$$

where $K_1(\cdot)$ denotes the modified Bessel function of second kind and first order [30, (9.6)].

The result below simplifies the second term of (20) and presents a more tractable upper bound SEP_{uu} for the SEP. The tightness of these bounds will be evaluated in Sec. V.

Result *4:* The SEP of optimal ARGTP relaying is upper bounded as follows:

$$\begin{aligned}\text{SEP} \leq \text{SEP}_u \leq \text{SEP}_{uu} &\triangleq I_1 + \frac{\text{SEP}_0}{\overline{\gamma}_{rd}\overline{\gamma}_{sr}} \int_0^\infty \Bigg[b_1^2 \text{Ei}\left(\frac{\mathcal{B}(\gamma_{sr})}{\overline{\gamma}_{rd}}\right) \\ &+ b_2\overline{\gamma}_{rd}\exp\left(-\frac{\mathcal{B}(\gamma_{sr})}{\overline{\gamma}_{rd}}\right) + 2b_1\sqrt{b_2\pi\overline{\gamma}_{rd}}\,\text{erfc}\left(\sqrt{\frac{\mathcal{B}(\gamma_{sr})}{\overline{\gamma}_{rd}}}\right)\Bigg] \\ &\times \mathcal{B}(\gamma_{sr})\exp\left(-\frac{\gamma_{sr}}{\overline{\gamma}_{sr}}\right) d\gamma_{sr}, \quad (22)\end{aligned}$$

where SEP_0 is given by (8), $b_1 = \frac{m^2\gamma_{sr}^2}{1+m^2\gamma_{sr}^2}, b_2 = \frac{m\gamma_{sr}}{\lambda(\gamma_{sr}+1)(1+m^2\gamma_{sr}^2)}$, $\text{erfc}(\cdot)$ denotes the complementary error function [29, (8.250.4)], and $\text{Ei}(\cdot)$ is the exponential integral $\text{Ei}(x) = \int_x^\infty \frac{e^{-t}}{t}\,dt$, for $x > 0$.

Proof: The derivation is given in Appendix D. ■

The upper bound SEP_{uu} can be accurately computed using Gauss-Laguerre quadrature [30] as:

$$\begin{aligned}\text{SEP}_{uu} \approx I_1 + \frac{\text{SEP}_0}{\overline{\gamma}_{rd}} \sum_{n=0}^{W} w_n \kappa_n \Bigg[\left(b_1^{(n)}\right)^2 \text{Ei}\left(\frac{\kappa_n}{\overline{\gamma}_{rd}}\right) \\ + b_2^{(n)}\overline{\gamma}_{rd} e^{-\frac{\kappa_n}{\overline{\gamma}_{rd}}} + 2b_1^{(n)}\sqrt{b_2^{(n)}\pi\overline{\gamma}_{rd}}\,\text{erfc}\left(\sqrt{\frac{\kappa_n}{\overline{\gamma}_{rd}}}\right)\Bigg], \quad (23)\end{aligned}$$

where w_n and a_n are the W weights and abscissas of Gauss-Laguerre quadrature, respectively, $\kappa_n = \frac{\lambda}{m}\left(1 + \frac{1}{a_n\overline{\gamma}_{sr}}\right)$, $b_1^{(n)} = \frac{a_n^2 m^2 \overline{\gamma}_{sr}^2}{(1+a_n^2 m^2 \overline{\gamma}_{sr}^2)}$, and $b_2^{(n)} = \frac{a_n m \overline{\gamma}_{sr}}{\lambda(a_n\overline{\gamma}_{sr}+1)(1+a_n^2 m^2\overline{\gamma}_{sr}^2)}$. We have found that at most $W = 6$ Gauss-Laguerre quadrature terms are sufficient to accurately compute SEP_{uu} up to 10^{-3}.

While SEP_{uu} is an upper bound, it is loose by 0.7 dB. We now present an alternate approximation that is accurate to within 0.1 dB, as shall be seen in Sec. V.

Result *5:*

$$\begin{aligned}\text{SEP} \approx \text{SEP}_{\text{apx}} = I_1 + \frac{\text{SEP}_0}{\overline{\gamma}_{rd}} \sum_{n=0}^{W_{\text{apx}}} w_n \\ \times \left[\kappa_n\left(\text{Ei}\left(\frac{\kappa_n}{\overline{\gamma}_{rd}}\right) - \text{Ei}\left(\frac{\kappa_n \exp(m\overline{\gamma}_{sr}a_n)}{\overline{\gamma}_{rd}}\right)\right)\right. \\ \left. + \overline{\gamma}_{rd}\exp\left(-\frac{\kappa_n\exp(m\overline{\gamma}_{sr}a_n)}{\overline{\gamma}_{rd}} - m\overline{\gamma}_{sr}a_n\right)\right]. \quad (24)\end{aligned}$$

Proof: The derivation is relegated to Appendix E. ■

We have found that at most $W_{\text{apx}} = 9$ terms are sufficient to accurately determine SEP_{apx} up to 10^{-3}.

A. Diversity Order Analysis

We now prove that the diversity order of optimal ARGTP relaying, which is denoted by d_{ARGTP}, is two for Rayleigh fading. Since fixed power relaying is a sub-optimal solution of the optimization problem in (9), we know that d_{ARGTP} is greater than or equal to the diversity order of fixed power relaying. As shown in [9], the diversity order of fixed power relaying for Rayleigh fading is 2. Thus, $d_{\text{ARGTP}} \geq 2$.[2]

Next, we analyze the diversity order of a lower bound on the SEP and use it to show that $d_{\text{ARGTP}} \leq 2$. We know that

$$\gamma_E = \gamma_{sd} + \frac{\beta_{\text{opt}}(\gamma_{sr},\gamma_{rd})\gamma_{sr}\gamma_{rd}}{\beta_{\text{opt}}(\gamma_{sr},\gamma_{rd})\gamma_{rd}+1} \leq \gamma_{sd} + \gamma_{sr}. \quad (25)$$

From the SEP expression in (4), it follows that

$$\text{SEP} \geq \frac{1}{\pi}\int_0^{\left(\frac{M-1}{M}\right)\pi} \mathbf{E}\left[\exp\left(-\left(\gamma_{sd}+\gamma_{sr}\right)\frac{m}{\sin^2\theta}\right)\right] d\theta.$$

Averaging over γ_{sr} and γ_{sd}, which are independent exponential random variables, we get

$$\text{SEP} \geq \frac{1}{\pi}\int_0^{\left(\frac{M-1}{M}\right)\pi} \frac{\sin^2\theta}{\left(\sin^2\theta + m\overline{\gamma}_{sr}\right)} \frac{\sin^2\theta}{\left(\sin^2\theta + m\overline{\gamma}_{sd}\right)} d\theta.$$

Using the inequality $\sin^2\theta \leq 1$ in the denominator terms above, we get

$$\text{SEP} \geq \frac{\frac{1}{\pi}\int_0^{\left(\frac{M-1}{M}\right)\pi} \sin^4\theta\, d\theta}{\left(1 + m\overline{\gamma}_{sr}\right)\left(1 + m\overline{\gamma}_{sd}\right)}. \quad (26)$$

It can be easily seen that the diversity order of the SEP lower bound in (26) is two, with the relay and direct links contributing equally. Hence, $d_{\text{ARGTP}} \leq 2$. Combining the results above, we get $d_{\text{ARGTP}} = 2$ for Rayleigh fading.

[2]Another relevant result that applies here is the diversity order analysis in [31] for fixed gain relaying for Nakagami-m fading. Specializing it to Rayleigh fading by setting $m = 1$ also yields $d_{\text{ARGTP}} \geq 2$.

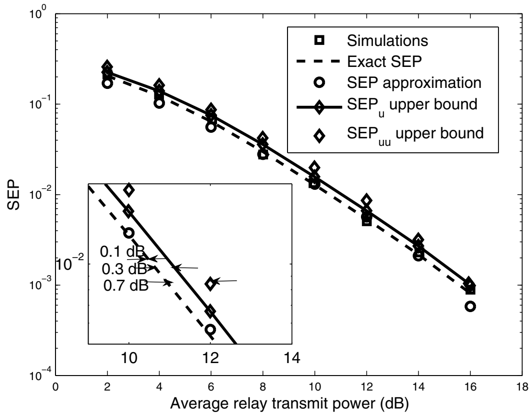

Fig. 4. SEP and its bounds as a function of average relay transmit power, $\overline{P}_r$ ($\overline{\gamma}_{sr} = \overline{\gamma}_{rd} = \overline{\gamma}_{sd} = P_s$, $P_s = \overline{P}_r$, and QPSK).

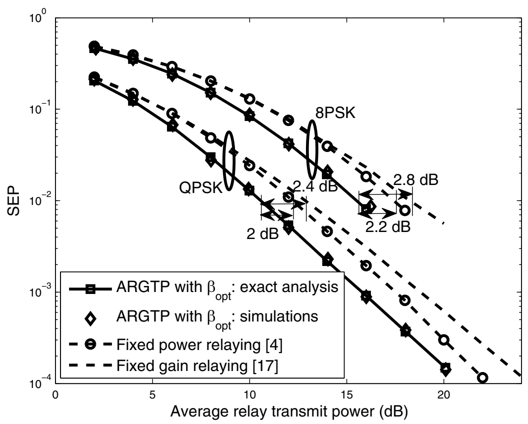

Fig. 5. Comparison of SEPs of AF relaying schemes as a function of average relay transmit power, $\overline{P}_r$ ($P_s = \overline{P}_r$ and $\overline{\gamma}_{sr} = \overline{\gamma}_{rd} = \overline{\gamma}_{sd} = P_s$).

V. Numerical Results and Discussion

We now verify the analytical results using Monte Carlo simulations that use up to 10^6 fading and noise realizations, and benchmark the performance of ARGTP relaying. In the simulations, we set $\mathbf{E}\left[|h_{sr}|^2\right] = 1$ and $\mathbf{E}\left[|h_{rd}|^2\right] = 1$. Recall that the additive noise power is normalized to unity. Given the focus of this paper on AF relaying, we compare ARGTP relaying with both fixed power and fixed gain relaying. We consider below two scenarios: (i) where the SD link is comparable in strength to the SR and RD links ($\mathbf{E}\left[|h_{sd}|^2\right] = 1$), and (ii) where the SD link is absent ($\mathbf{E}\left[|h_{sd}|^2\right] = 0$).

A. With SD link

Figure 4 plots the SEP expression in (18) as a function of the average relay transmit power for QPSK. Also plotted for reference are the results from Monte Carlo simulations, the SEP upper bounds SEP$_u$ in (20) and SEP$_{uu}$ in (23), and the SEP approximation SEP$_{\text{apx}}$ in (24). We see that SEP$_u$ and SEP$_{uu}$ are within 0.3 dB and 0.7 dB, respectively, at an SEP of 10^{-2}. However, the approximation is accurate to within 0.1 dB.

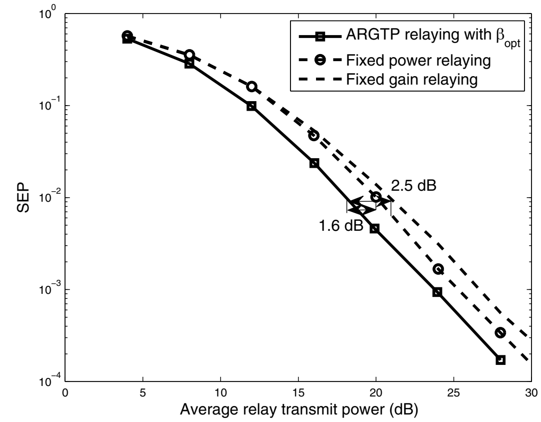

Fig. 6. MQAM: Comparison of SEPs of AF relaying schemes as a function of average relay transmit power, $\overline{P}_r$ ($\overline{\gamma}_{sr} = \overline{\gamma}_{rd} = \overline{\gamma}_{sd} = P_s$, $P_s = \overline{P}_r$, and 16QAM).

Figure 5 plots the SEP as a function of the average relay transmit power for QPSK and 8PSK. It compares the SEPs of ARGTP relaying, fixed gain relaying, and fixed power relaying when $P_s = \overline{P}_r$. We see that ARGTP relaying outperforms both fixed gain and fixed power relaying. For example, for QPSK, it requires 2.0 dB (36.9%) less power than fixed power relaying and 2.4 dB (42.5%) less power than fixed gain relaying at an SEP of 10^{-2}. For 8PSK, the power savings increase to 2.2 dB (39.7%) over fixed power relaying and 2.8 dB (47.5%) over fixed gain relaying. At larger SNRs, the SEP curves of ARGTP relaying and fixed power relaying become parallel to each other, which shows that their diversity orders become the same (two). However, fixed gain relaying requires considerably larger SNRs to reach the asymptotic regime in which its diversity order is two [26], [32].

Similarly, Figure 6 benchmarks the SEP of ARGTP relaying with fixed gain and fixed power relaying for 16QAM. We again see that ARGTP relaying markedly outperforms both fixed gain and fixed power relaying. For example, it requires 1.6 dB (30.8%) less power than fixed power relaying and requires 2.5 dB (43.8%) less power than fixed gain relaying at an SEP of 10^{-2}.

B. Without SD link

Figure 7 plots the SEP of ARGTP relaying as a function of the average relay transmit power for QPSK. Also shown are the corresponding SEP curves for fixed gain and fixed power relaying. Now ARGTP relaying outperforms both fixed gain and fixed power AF relaying by an even larger margin. For example, the savings are 4.3 dB (62.8%) over fixed power relaying and 7.7 dB (83.0%) over fixed gain relaying at an SEP of 2×10^{-2}. This is because in the absence of the direct SD link, the relay influences the signal quality at the destination much more.

Figure 8 compares the finite-SNR diversity orders of ARGTP relaying, fixed power relaying, and fixed gain relaying. The finite-SNR diversity order is defined as the negative slope of the error probability curve at a given SNR [18], [19]. It quantifies how well the cooperation protocol harnesses

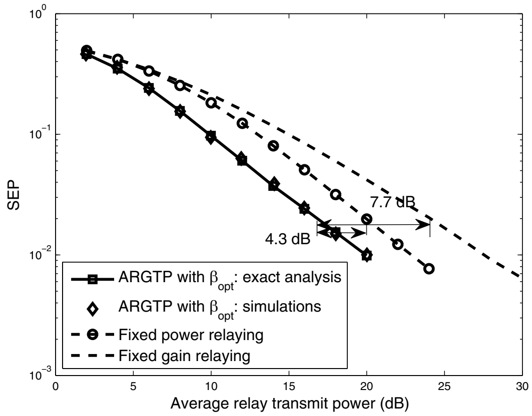

Fig. 7. Without SD link: Comparison of SEPs of AF relaying schemes as a function of average relay transmit power, $\overline{P}_r$ ($\overline{\gamma}_{sr} = \overline{\gamma}_{rd} = P_s$, $\overline{\gamma}_{sd} = 0$, and QPSK).

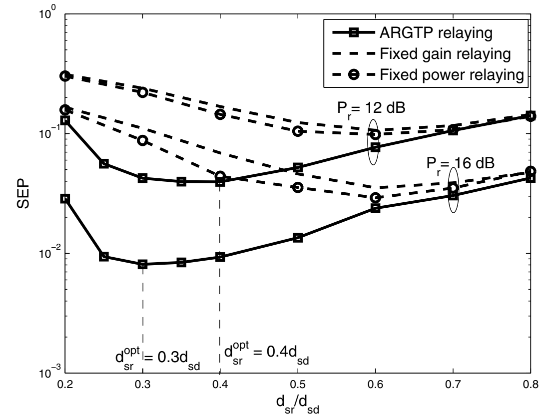

Fig. 9. Optimal relay placement: SEP of ARGTP relaying, fixed gain relaying, and fixed power relaying as a function of d_{sr} ($P_s = \overline{P}_r$, $\eta = 4$, and QPSK).

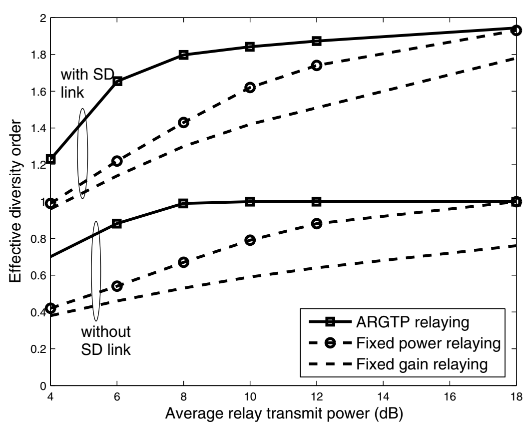

Fig. 8. Finite-SNR diversity order comparison of AF relaying schemes as a function of average relay transmit power, $\overline{P}_r$ (QPSK; with SD link: $\overline{\gamma}_{sr} = \overline{\gamma}_{rd} = \overline{\gamma}_{sd} = P_s$; without SD link: $\overline{\gamma}_{sr} = \overline{\gamma}_{rd} = P_s$, and $\overline{\gamma}_{sd} = 0$).

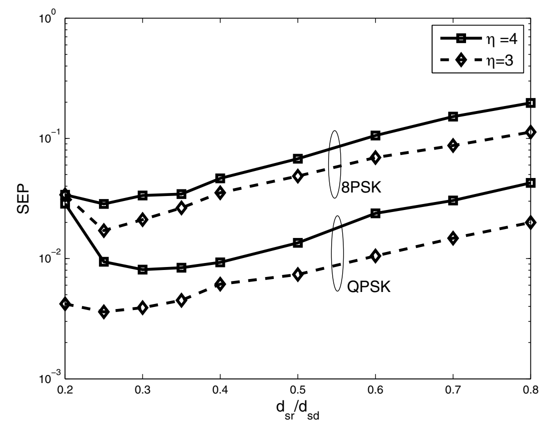

Fig. 10. Optimal relay placement: SEP of ARGTP relaying as a function of d_{sr} ($P_s = \overline{P}_r = 16$ dB).

spatial diversity at a given SNR and not just at asymptotically large SNRs. The figure shows that while the three schemes have an asymptotic diversity order of two when the SD link is present and one when it is absent (cf. Sec. IV-A) for Rayleigh fading, the behavior of their finite-SNR diversity orders is different. The finite-SNR diversity order of ARGTP relaying is greater than the benchmark schemes for SNRs up to 16 dB. This addresses one problem of fixed power and fixed gain relaying, which require large SNRs to achieve full diversity [32].

C. Optimal Relay Placement

We now study the optimum placement of an ARGTP relay and contrast it with that for the conventional schemes. Let d_{sr}, d_{rd}, and d_{sd} denote the distance between S and R, R and D, and S and D, respectively. For simplicity, the relay is placed on the line segment connecting the source and the destination. Thus, $d_{sd} = d_{sr} + d_{rd}$. As before, $\mathbf{E}\left[|h_{sr}|^2\right] = \mathbf{E}\left[|h_{rd}|^2\right] = \mathbf{E}\left[|h_{sd}|^2\right] = 1$. After accounting for path loss, the average SNRs of the SR, RD, and SD links are given by $\overline{\gamma}_{sr} = P_s\left(\frac{d_0}{d_{sr}}\right)^{\eta}$, $\overline{\gamma}_{rd} = \overline{P}_r\left(\frac{d_0}{d_{rd}}\right)^{\eta}$, and $\overline{\gamma}_{sd} = P_s\left(\frac{d_0}{d_{sd}}\right)^{\eta}$, where η is the path loss exponent and d_0 is a reference distance. In the results below, we set $\eta = 4$.

Figure 9 plots the SEP as a function of d_{sr} for all the three AF relaying schemes. For ARGTP relaying, the optimal relay position is $0.4d_{sd}$ from the source when $\overline{P}_r = P_s = 12$ dB and is $0.3d_{sd}$ when $\overline{P}_r = P_s = 16$ dB. However, it is $0.6d_{sd}$ for both fixed power and fixed gain relaying at both the above power settings. Thus, the optimal relay location for ARGTP relaying is closer to the source than for the other benchmark relaying schemes. This can be intuitively understood as follows. When the relay is closer to the destination, the odds of the RD channel gain being larger are high. As discussed in Sec. III-B, the amplification of the signal by the relay in this scenario provides limited benefit. Similarly, when the relay is very close to the source, the odds of the RD channel gain being very small are high. Therefore, the relay, having been turned off, provides no benefit most of the time. It is for this reason that the SEP first falls sharply as d_{sr} increases, reaches a minimum value $d_{sr} = 0.3d_{sd}$, and then increases gradually as d_{sr} increases further. Also notice that the SEP is always the lowest for ARGTP relaying.

Figure 10 plots the SEP as a function of the relay location for different values of the constellation size M and path loss exponent η. We see that the optimal relay location is insensitive to both η and M.

VI. Conclusions

We considered ARGTP relaying, in which, the relay optimally adapts its instantaneous transmit power and gain as a function of the channel gains of the links incident on the relay. We derived the SEP-optimal AF relay gain and an analytically tractable, insightful and implementable upper bound and lower bound for it. We then analyzed its SEP.

The numerical results for MPSK and MQAM showed that ARGTP relaying, which optimally exploits the knowledge of the local channel gains at the relay, markedly outperforms both fixed power and fixed gain relaying. We also saw the optimal location of an ARGTP relay is closer to the source than for a fixed gain or fixed power relay.

The results motivate the use of ARGTP relaying in cooperative relay networks. Interesting problems for future work include introduction of a peak power constraint, including imperfect channel state information in our model, and its extension to multi-hop and multi-relay systems. For example, in a two-hop system with multiple AF relays, the selected relay can follow the relay gain rule in (13), with the relay that maximizes the SNR at the destination being selected.

Appendix

A. Proof of Result 1

Let $f(x) \triangleq \exp\left(-\frac{m\gamma_{sr}\gamma_{rd}}{\gamma_{rd}+x^{-1}}\right)$. Partially differentiating $f(x)$ with respect to x, we get

$$\frac{\partial f(x)}{\partial x} = -\exp\left(-\frac{m\gamma_{sr}\gamma_{rd}}{\gamma_{rd}+x^{-1}}\right)\frac{m\gamma_{sr}\gamma_{rd}}{(x\gamma_{rd}+1)^2}.$$

Therefore, $\frac{\partial f(x)}{\partial x} \leq 0$, for all $x \geq 0$. The second derivative of $f(x)$ is

$$\frac{\partial^2 f(x)}{\partial x^2} = \exp\left(-\frac{m\gamma_{sr}\gamma_{rd}}{\gamma_{rd}+x^{-1}}\right)\left(\frac{m^2\gamma_{sr}^2\gamma_{rd}^2}{(x\gamma_{rd}+1)^2}+\frac{2m\gamma_{sr}\gamma_{rd}^2}{(x\gamma_{rd}+1)^3}\right).$$

Thus, $\frac{\partial^2 f(x)}{\partial x^2} \geq 0$, for all $x \geq 0$. Hence, $f(x)$ is a convex function of x, for $x \geq 0$, and so is its expectation in (9). Further, since $\beta(\gamma_{sr},\gamma_{rd})(\gamma_{sr}+1)$ is a linear function of $\beta(\gamma_{sr},\gamma_{rd})$, the average power constraint is also convex. Therefore, the problem is equivalent to minimizing $\mathbf{E}\left[L_\lambda(\beta(\gamma_{sr},\gamma_{rd}))\right]$, where

$$L_\lambda(x) \triangleq \exp\left(-\frac{m\gamma_{sr}\gamma_{rd}}{\gamma_{rd}+x^{-1}}\right) + \lambda\left(x\left(\gamma_{sr}+1\right)-1\right). \quad (27)$$

Since this is now an unconstrained optimization, minimizing $\mathbf{E}\left[L_\lambda(\beta(\gamma_{sr},\gamma_{rd}))\right]$ is equivalent to minimizing $L_\lambda(\beta(\gamma_{sr},\gamma_{rd}))$ for each value of γ_{sr} and γ_{rd}. Formally, the optimization problem for each γ_{sr}, γ_{rd} can be stated as:

$$\begin{aligned} \min_x \quad & L_\lambda(x) \\ \text{s. t.} \quad & x \geq 0. \end{aligned} \quad (28)$$

Since $L_\lambda(x)$ is convex in x, the optimal value of x is unique. It is the non-negative solution of

$$\frac{\partial L_\lambda(x)}{\partial x} = -\exp\left(-\frac{m\gamma_{sr}\gamma_{rd}}{\gamma_{rd}+x^{-1}}\right)\frac{m\gamma_{sr}\gamma_{rd}}{(\gamma_{rd}x+1)^2} + \lambda(\gamma_{sr}+1) = 0, \quad (29)$$

if it exists, and is 0, otherwise. Let the non-negative solution be denoted by $x_0(\gamma_{sr},\gamma_{rd})$. Simplifying (29) results in (12). The boundary of the region in which $\beta_{\text{opt}}(\gamma_{sr},\gamma_{rd})$ is 0 is obtained by substituting $x=0$ in (12). Further, it can be verified that $\beta_{\text{opt}}(\gamma_{sr},\gamma_{rd}) = 0$, for all $\gamma_{rd} < \mathcal{B}(\gamma_{sr})$.

B. Proof of Result 2

Since $\beta_{\text{opt}}(\gamma_{sr},\gamma_{rd})$ satisfies (12), by rearranging its terms, we get the following for $\gamma_{rd} \geq \mathcal{B}(\gamma_{sr})$:

$$(\beta_{\text{opt}}(\gamma_{sr},\gamma_{rd})\gamma_{rd}+1)^2 = \frac{m\gamma_{sr}\gamma_{rd}}{\lambda(\gamma_{sr}+1)}\exp\left(-\frac{m\gamma_{sr}\gamma_{rd}}{\gamma_{rd}+\beta_{\text{opt}}(\gamma_{sr},\gamma_{rd})^{-1}}\right). \quad (30)$$

Since $\beta_{\text{opt}}(\gamma_{sr},\gamma_{rd}) \geq 0$, it follows that $\frac{\gamma_{rd}}{\gamma_{rd}+\beta_{\text{opt}}(\gamma_{sr},\gamma_{rd})^{-1}} \leq 1$. Hence,

$$\exp\left(-\frac{m\gamma_{sr}\gamma_{rd}}{\gamma_{rd}+\beta_{\text{opt}}(\gamma_{sr},\gamma_{rd})^{-1}}\right) \geq \exp\left(-m\gamma_{sr}\right). \quad (31)$$

Substituting (31) in (30) yields

$$(\beta_{\text{opt}}(\gamma_{sr},\gamma_{rd})\gamma_{rd}+1)^2 \geq \frac{m\gamma_{sr}\gamma_{rd}}{\lambda\left(\gamma_{sr}+1\right)}\exp\left(-m\gamma_{sr}\right). \quad (32)$$

Taking square root on both sides and simplifying, we get

$$\begin{aligned} \beta_{\text{opt}}(\gamma_{sr},\gamma_{rd}) &\geq \frac{1}{\gamma_{rd}}\left(\sqrt{\frac{m\gamma_{sr}\gamma_{rd}}{\lambda\left(\gamma_{sr}+1\right)}}\exp\left(-\frac{m\gamma_{sr}}{2}\right)-1\right), \\ &\triangleq \beta_l(\gamma_{sr},\gamma_{rd}). \end{aligned} \quad (33)$$

Since $\beta_{\text{opt}}(\gamma_{sr},\gamma_{rd}) \geq 0$, the above lower bound can be tightened by ensuring $\beta_l(\gamma_{sr},\gamma_{rd}) \geq 0$. From (33), we see that this occurs so long as $\sqrt{\frac{m\gamma_{sr}\gamma_{rd}}{\lambda(\gamma_{sr}+1)}}\exp\left(-\frac{m\gamma_{sr}}{2}\right) \geq 1$. From the definition of $\mathcal{B}(\gamma_{sr})$ in (14), this is equivalent to the condition

$$\gamma_{rd} \geq \mathcal{B}(\gamma_{sr})\exp\left(m\gamma_{sr}\right). \quad (34)$$

Combining the results in (33) and (34) yields (15).

C. Proof of Result 3

The following simple inequality shall help us derive a tractable upper bound for β_{opt}: $\exp(x) \geq 1+x^2$, for $x \geq 0$. Substituting it in (12), we get, for $\gamma_{rd} \geq \mathcal{B}(\gamma_{sr})$,

$$\phi(x) \geq (x\gamma_{rd}+1)^2\left(1+\frac{a^2x^2}{(x\gamma_{rd}+1)^2}\right) - \frac{a}{\lambda\left(\gamma_{sr}+1\right)},$$

where $a \triangleq m\gamma_{sr}\gamma_{rd}$. Simplifying further, we get

$$\phi(x) \geq (x\gamma_{rd}+1)^2 + a^2x^2 - \frac{a}{\lambda\left(\gamma_{sr}+1\right)}. \quad (35)$$

Expanding the right side of the above inequality and rearranging terms yields the following quadratic form in x:

$$\phi(x) \geq \left(\gamma_{rd}^2+a^2\right)x^2 + 2\gamma_{rd}x + 1 - \frac{a}{\lambda\left(\gamma_{sr}+1\right)} \triangleq \Omega(x).$$

It can be easily verified that $\Omega(x)$ has exactly one positive root when $\gamma_{rd} \geq \mathcal{B}(\gamma_{sr})$. It is nothing but $\beta_u(\gamma_{sr},\gamma_{rd})$, which is given in (17). Since $\phi(x) \geq \Omega(x)$ and since both are convex for $x \geq 0$, it can be shown that $\beta_{\text{opt}}(\gamma_{sr},\gamma_{rd}) \leq \beta_u(\gamma_{sr},\gamma_{rd})$.

D. Derivation of Result 4

Since $\beta_{\text{opt}} \leq \beta_u$, it follows from (17) that, for $\gamma_{rd} \geq \mathcal{B}(\gamma_{sr})$,

$$\beta_{\text{opt}}(\gamma_{sr}, \gamma_{rd})\gamma_{rd} + 1 \leq \beta_u(\gamma_{sr}, \gamma_{rd})\gamma_{rd} + 1 \leq \frac{\gamma_{sr}^2 m^2 + \sqrt{\frac{\gamma_{sr}\gamma_{rd} m}{\lambda(\gamma_{sr}+1)}\left(1+\gamma_{sr}^2 m^2\right)}}{1+\gamma_{sr}^2 m^2}.$$

Substituting this and (21) in (20), we get the following upper bound for SEP_u:

$$\text{SEP}_u \leq \text{SEP}_{uu} = I_1 + \frac{\text{SEP}_0}{\overline{\gamma}_{sr}\overline{\gamma}_{rd}} \int_0^\infty \int_{\mathcal{B}(\gamma_{sr})}^\infty \frac{\lambda\left(\gamma_{sr}+1\right)\left(b_1+\sqrt{b_2\gamma_{rd}}\right)^2}{\gamma_{sr}\gamma_{rd} m} \times \exp\left(-\frac{\gamma_{sr}}{\overline{\gamma}_{sr}}\right)\exp\left(-\frac{\gamma_{rd}}{\overline{\gamma}_{rd}}\right) d\gamma_{sr}\, d\gamma_{rd}, \quad (36)$$

where b_1 and b_2 are as defined in the result statement. The double integral term in (36), which is denoted by I_2, can be recast as

$$I_2 = \frac{\text{SEP}_0}{\overline{\gamma}_{sr}\overline{\gamma}_{rd}} \int_0^\infty \mathcal{B}(\gamma_{sr}) \exp\left(-\frac{\gamma_{sr}}{\overline{\gamma}_{sr}}\right) I_2^{\text{in}}(\gamma_{sr})\, d\gamma_{sr}, \quad (37)$$

where $I_2^{\text{in}} \triangleq \int_{\mathcal{B}(\gamma_{sr})}^\infty \frac{b_1^2 + b_2\gamma_{rd} + 2b_1\sqrt{b_2\gamma_{rd}}}{\gamma_{rd}} \exp\left(-\frac{\gamma_{rd}}{\overline{\gamma}_{rd}}\right) d\gamma_{rd}$. Further, I_2^{in} can be split as

$$I_2^{\text{in}} = b_1^2\varphi_1 + b_2\varphi_2 + 2b_1\sqrt{b_2}\varphi_3, \quad (38)$$

where $\varphi_1 \triangleq \int_{\mathcal{B}(\gamma_{sr})}^\infty \frac{1}{\gamma_{rd}} \exp\left(-\frac{\gamma_{rd}}{\overline{\gamma}_{rd}}\right) d\gamma_{rd} = \text{Ei}\left(\frac{\mathcal{B}(\gamma_{sr})}{\overline{\gamma}_{rd}}\right)$, $\varphi_2 \triangleq \int_{\mathcal{B}(\gamma_{sr})}^\infty \exp\left(-\frac{\gamma_{rd}}{\overline{\gamma}_{rd}}\right) d\gamma_{rd} = \overline{\gamma}_{rd}\exp\left(-\frac{\mathcal{B}(\gamma_{sr})}{\overline{\gamma}_{rd}}\right)$, and $\varphi_3 \triangleq \int_{\mathcal{B}(\gamma_{sr})}^\infty \frac{1}{\sqrt{\gamma_{rd}}} \exp\left(-\frac{\gamma_{rd}}{\overline{\gamma}_{rd}}\right) d\gamma_{rd} = \sqrt{\pi\overline{\gamma}_{rd}}\,\text{erfc}\left(\sqrt{\frac{\mathcal{B}(\gamma_{sr})}{\overline{\gamma}_{rd}}}\right)$. Substituting all the above expressions in (36) yields the desired expression in (22).

E. Derivation of Result 5

The derivation below uses the properties of β_{opt} to derive a lower bound for SEP_u, which we denote by SEP_{apx}. The goal is to arrive at a simpler expression that is in the form of a single integral, unlike the double integral form of the expression in (20). Note that since SEP_{apx} lower bounds SEP_u, it is neither a lower nor an upper bound for the exact SEP. However, it is useful as an approximation for the SEP. Recall from (20) that the second term in the expression for SEP_u is

$$J_2 = \frac{\text{SEP}_0}{\overline{\gamma}_{sr}\overline{\gamma}_{rd}} \int_0^\infty \int_{\mathcal{B}(\gamma_{sr})}^\infty \frac{\lambda(\gamma_{sr}+1)(\beta_{\text{opt}}(\gamma_{sr}, \gamma_{rd})\gamma_{rd}+1)^2}{m\gamma_{sr}\gamma_{rd}} \times \exp\left(-\frac{\gamma_{sr}}{\overline{\gamma}_{sr}}\right)\exp\left(-\frac{\gamma_{rd}}{\overline{\gamma}_{rd}}\right) d\gamma_{sr}\, d\gamma_{rd}. \quad (39)$$

Replacing β_{opt} with its lower bound β_l in the above equation yields a lower bound for J_2. Further, since $\beta_l(\gamma_{sr}, \gamma_{rd}) = 0$, for $\gamma_{rd} \leq \mathcal{B}(\gamma_{sr})\exp\left(m\gamma_{sr}\right) = \mathcal{B}_1(\gamma_{sr})$, we get

$$J_2 \geq \frac{\text{SEP}_0}{\overline{\gamma}_{sr}\overline{\gamma}_{rd}} \int_0^\infty \int_{\mathcal{B}(\gamma_{sr})}^{\mathcal{B}_1(\gamma_{sr})} \frac{\lambda(\gamma_{sr}+1)}{m\gamma_{sr}\gamma_{rd}} \exp\left(-\frac{\gamma_{sr}}{\overline{\gamma}_{sr}}\right) \times \exp\left(-\frac{\gamma_{rd}}{\overline{\gamma}_{rd}}\right) d\gamma_{sr}\, d\gamma_{rd} + \frac{\text{SEP}_0}{\overline{\gamma}_{sr}\overline{\gamma}_{rd}} \int_0^\infty \int_{\mathcal{B}_1(\gamma_{sr})}^\infty \frac{\lambda(\gamma_{sr}+1)(\beta_l(\gamma_{sr}, \gamma_{rd})\gamma_{rd}+1)^2}{m\gamma_{sr}\gamma_{rd}} \times \exp\left(-\frac{\gamma_{sr}}{\overline{\gamma}_{sr}}\right)\exp\left(-\frac{\gamma_{rd}}{\overline{\gamma}_{rd}}\right) d\gamma_{sr}\, d\gamma_{rd}. \quad (40)$$

However, from (15), we know that, for $\gamma_{rd} \geq \mathcal{B}_1(\gamma_{sr})$,

$$\frac{\lambda\left(\gamma_{sr}+1\right)\left(\beta_l(\gamma_{sr}, \gamma_{rd})\gamma_{rd}+1\right)^2}{m\gamma_{sr}\gamma_{rd}} = \exp\left(-m\gamma_{sr}\right).$$

Substituting this in (40), we get

$$J_2 \geq \frac{\text{SEP}_0}{\overline{\gamma}_{sr}\overline{\gamma}_{rd}} \int_0^\infty \int_{\mathcal{B}(\gamma_{sr})}^{\mathcal{B}_1(\gamma_{sr})} \frac{\lambda\left(\gamma_{sr}+1\right)}{m\gamma_{sr}\gamma_{rd}} \exp\left(-\frac{\gamma_{sr}}{\overline{\gamma}_{sr}}\right) \times \exp\left(-\frac{\gamma_{rd}}{\overline{\gamma}_{rd}}\right) d\gamma_{sr}\, d\gamma_{rd} + \frac{\text{SEP}_0}{\overline{\gamma}_{sr}\overline{\gamma}_{rd}} \int_0^\infty \int_{\mathcal{B}_1(\gamma_{sr})}^\infty \exp\left(-m\gamma_{sr}\right)\exp\left(-\frac{\gamma_{sr}}{\overline{\gamma}_{sr}}\right) \times \exp\left(-\frac{\gamma_{rd}}{\overline{\gamma}_{rd}}\right) d\gamma_{sr}\, d\gamma_{rd}. \quad (41)$$

Let the first term in (41) above be denoted by L_1. By noting that $\frac{\lambda(\gamma_{sr}+1)}{m\gamma_{sr}\gamma_{rd}} = \frac{\mathcal{B}(\gamma_{sr})}{\gamma_{rd}}$ (cf. (14)), L_1 can be recast as

$$L_1 = \frac{\text{SEP}_0}{\overline{\gamma}_{sr}\overline{\gamma}_{rd}} \int_0^\infty \mathcal{B}(\gamma_{sr}) \exp\left(-\frac{\gamma_{sr}}{\overline{\gamma}_{sr}}\right) L_1^{\text{in}}(\gamma_{sr})\, d\gamma_{sr}, \quad (42)$$

where

$$L_1^{\text{in}} = \int_{\mathcal{B}(\gamma_{sr})}^{\mathcal{B}_1(\gamma_{sr})} \frac{1}{\gamma_{rd}} \exp\left(-\frac{\gamma_{rd}}{\overline{\gamma}_{rd}}\right) d\gamma_{rd}, = \text{Ei}\left(\frac{\mathcal{B}(\gamma_{sr})}{\overline{\gamma}_{rd}}\right) - \text{Ei}\left(\frac{\mathcal{B}_1(\gamma_{sr})}{\overline{\gamma}_{rd}}\right). \quad (43)$$

The second term in (41), which we denote by L_2, can be rewritten as

$$L_2 = \int_0^\infty \exp\left(-m\gamma_{sr}\right)\exp\left(-\frac{\gamma_{sr}}{\overline{\gamma}_{sr}}\right) L_2^{\text{in}}(\gamma_{sr})\, d\gamma_{sr}, \quad (44)$$

where

$$L_2^{\text{in}} \triangleq \int_{\mathcal{B}_1(\gamma_{sr})}^\infty \exp\left(-\frac{\gamma_{rd}}{\overline{\gamma}_{rd}}\right) d\gamma_{rd} = \overline{\gamma}_{rd}\exp\left(-\frac{\mathcal{B}_1(\gamma_{sr})}{\overline{\gamma}_{rd}}\right).$$

Thus, L_2 simplifies to

$$L_2 = \overline{\gamma}_{rd}\int_0^\infty \exp(-m\gamma_{sr})\exp\left(-\frac{\gamma_{sr}}{\overline{\gamma}_{sr}}\right)\exp\left(-\frac{\mathcal{B}_1(\gamma_{sr})}{\overline{\gamma}_{rd}}\right) d\gamma_{sr}. \quad (45)$$

Upon substituting (21), (42), (43), and (45) in (20), we get

$$\text{SEP}_u \geq \text{SEP}_{\text{apx}} = I_1 + \frac{\text{SEP}_0}{\overline{\gamma}_{rd}\overline{\gamma}_{sr}} \int_0^\infty \left[\mathcal{B}(\gamma_{sr})\left(\text{Ei}\left(\frac{\mathcal{B}(\gamma_{sr})}{\overline{\gamma}_{rd}}\right) - \text{Ei}\left(\frac{\mathcal{B}_1(\gamma_{sr})}{\overline{\gamma}_{rd}}\right)\right) + \overline{\gamma}_{rd}\exp\left(-\frac{\mathcal{B}_1(\gamma_{sr})}{\overline{\gamma}_{rd}} - m\gamma_{sr}\right)\right]\exp\left(-\frac{\gamma_{sr}}{\overline{\gamma}_{sr}}\right) d\gamma_{sr}. \quad (46)$$

The desired expression in (24) is obtained by using Gauss-Laguerre quadrature.

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