Problems: With something to think about on each question.

1. Give an approximation for \( E[N|H_0] \) analogous to the approximation \( E[N|H_1] \) obtained in class. Explain why the two expressions are different (both numerator and denominator).

2. Suppose that the target false alarm rate and miss probabilities are identical and given by \( \varepsilon \). Assume that the two hypotheses have priors \( \pi_0 \) and \( \pi_1 = 1 - \pi_0 \). Give an approximation for the limiting value
\[
\lim_{\varepsilon \to 0} \frac{E[N]}{\log(1/\varepsilon)}.
\]
This not only tells how the number of samples grows as \( \varepsilon \) shrinks, but also gives the proportionality constant.

3. Consider the following sequential detection problem.
\[
H_0 : Y_k = Z_k, \quad k = 1, 2, \ldots
\text{versus}
H_1 : Y_k = \theta + Z_k, \quad k = 1, 2, \ldots, \quad \theta > 0.
\]
where \( Z_k \) are iid \( N(0, 1) \). Find \( D(P_1||P_0) \) and \( D(P_0||P_1) \). Does the answer surprise you? Assuming that the false alarm rate and miss probability are \( \varepsilon \) give an expression for the (approximate) expected number of samples for a decision.

4. For the hypothesis testing problem above, take \( \theta = 1 \). But consider a fixed number of samples \( n_0 \). Using an expression relating power and size of a fixed sample test, describe how to obtain \( n_0 \) so that the false alarm rate and miss probability are both \( \varepsilon \). For small \( \varepsilon \), say \( \varepsilon = 0.01 \) compare \( n_0 \) of this problem and \( E[N|H_j] \) of the previous problem. Which is better?

   Step 1: In class we considered the case of uniform costs. Consider the more general case where \( C_{01} = w \), \( C_{10} = 1 - w \), and \( C_{11} = C_{00} = 0 \). Let \( c \) be the cost per sample. For a fixed \( w \), verify that the optimal sequential decision rule is an SPRT test with \( \pi_L(c, w) \leq \pi_U(c, w) \) as the lower and upper thresholds, respectively, on the posterior probability.

6. Step 2: Let the prior \( \pi_1 \) satisfy \( \pi_L(c, w) \leq \pi_1 \leq \pi_U(c, w) \). Identify the threshold \( A \) and \( B \) for the likelihood ratio in the SPRT\((A, B)\).

7. Step 3: For a fixed \( w \), assume that \( \pi_L(c, w) \) and \( \pi_U(c, w) \) are continuous, and further assume that \( \lim_{c \to 0} \pi_L(c, w) = 0 = 1 - \lim_{c \to 0} \pi_U(c, w) \). Is this reasonable? Fix any \( \varepsilon > 0 \) and \( 0 < A \leq B < +\infty \), show that there exist \( (a) \pi_1, c, w \) having \( 0 < \pi_1 < \varepsilon \) and \( A, B \) as given by the formulae in the previous step.

8. Step 4: Consider now an SPRT \((\phi, \delta)\) and any other test \((\phi', \delta')\) such that
\[
P_F(\phi', \delta') \leq P_F(\phi, \delta) \quad \text{and} \quad P_M(\phi', \delta') \leq P_M(\phi, \delta).
\]
Using the previous steps to argue that \( E_0[N(\phi')] \geq E_0[N(\phi)] \). Outline the steps to prove the inequality under \( H_1 \).

Problem Set 4-1