Problems:

1. For the Kalman-Bucy filter, verify that the innovations sequence \( \{ I_n, n \geq 0 \} \), where \( I_n = Y_n - H_n \hat{X}_{n|n-1} \), is an independent sequence (under Gaussian assumptions).

2. Exercise 2 in Section V.E.

3. Exercise 5 in Section V.E.

4. Exercise 11 in Section V.E.

5. Consider the model \( Y = sX + Z \) where \( s \in \mathbb{R}^N \) is some known signature vector, \( X \) is \( \pm 1 \) with equal probability, and \( Z \) has mean 0 and covariance \( \Sigma \). The goal is to estimate \( X \) having observed \( Y \) with estimates of the form \( \hat{X} = h^T Y \). Consider now the following iterative procedure with training where you are given \( \{(X_i, Y_i), i = 1, 2, \ldots, r\} \) for \( r \) training instants. Define \( \phi(h) := \mathbb{E}[(X - h^T Y)^2] \) and fix \( \mu > 0 \). Start with a candidate \( h(0) \), and update as follows:

\[
    h(k) = h(k-1) - \mu \nabla \phi(h(k-1)), \quad k \geq 1.
\]

Towards which point \( h^* \) are these iterates taking you? Give a heuristic justification for your answer.

6. Let \( Y_n = X_n + Z_n \) where \( -\infty < n < \infty \). Assume that \( \{X_n\} \) and \( \{Z_n\} \) are uncorrelated, zero mean, and have power spectral densities \( \phi_X(\omega) \) and \( \phi_Z(\omega) \), respectively. Find the transfer function for the noncausal Wiener-Kolmogorov filter in terms of \( \phi_X(\omega) \) and \( \phi_Z(\omega) \). Give an expression for the (linear) minimum mean squared error in terms of these power spectral densities.

7. Let \( \tilde{X}_t \) be the best linear noncausal estimate of \( X_t \) given the observations \( \{Y_n, -\infty < n < \infty\} \).

Let \( \hat{X}_t \) denote the best linear causal estimate of \( X_t \) given the observations \( \{Y_n, -\infty < n \leq t\} \).

Argue that \( \hat{X}_t \) is the linear MMSE of \( \tilde{X}_t \) given \( \{Y_n, -\infty < n \leq t\} \), and interpret.