1. (FDMA vs $\mathcal{C}_{\text{MAC}}$)

   a) Consider a single user communication system having a passband of $[f_c - B/2, f_c + B/2]$ Hz and having an average power constraint of $P$ Joules/sec. What is the capacity $C(P, B)$?
   
   b) Consider the same system as above, but the passband is $[f_c - B/2, f_c]$ Hz. What is the capacity, $C(P, B/2)$?

   c) Show that $C(P, B/2) \leq C(P, B)$ by proving $C(P, B)$ is an increasing function of $B$.

   d) Consider a two user FDMA system having an average power constraint $(P, P)$. User 1 uses $[f_c - B/2, f_c]$ and user 2 uses $[f_c, f_c + B/2]$. Show that $C(P, B/2) + C(P, B/2) = C(2P, B)$.

   e) Is there any other point on the dominant facet $(R_1 + R_2 = C(2P, B))$ that is attained by FDMA.

   (Hint: A passband $B$ requires $B$ complex dimensions per second or $2B$ real dimensions per second)

2. (Frequency typicality) For every $\delta > 0$, the following hold for all sufficiently large $n$. Prove them (Notation is as given in lecture notes).

   a) $\Pr \left\{ Z_n^1 \in T_{\delta}^{(n)} \right\} \geq 1 - \delta$ and therefore $\Pr \left\{ Z_n^1 \in T_{\delta}^{(n)}(Z_A) \right\} \geq 1 - \delta$.

   b) $z_A^1 \in T_{\delta}^{(n)}(Z_A) \mapsto \frac{1}{n} \log p_{Z_A}(z_A^1) + H(Z_A) \leq \delta$.

   c) $(z_A^1, z_B^1) \in T_{\delta}^{(n)}(Z_{A\cup B})$, $A \cap B = \emptyset \implies \frac{1}{n} \log p_{Z_A|Z_B}(z_A^1|z_B^1) + H(Z_A|Z_B) \leq 2\delta$.

   d) $(1 - \delta)2^{nH(Z_A) - n\delta} \leq \left| T_{\delta}^{(n)}(Z_A) \right| \leq 2^{nH(Z_A) + n\delta}$ so that $\left| T_{\delta}^{(n)}(Z_A) \right| \approx 2^{nH(Z_A) + \pm 2n\delta}$.

   e) $\widetilde{Z}_{[m]} = \sim p_{Z_A P_B|Z_A} p_{Z_C|Z_A}$, $A \cup B \cup C = [m]$, $A \cap B = B \cap C = C \cap A = \emptyset$, $\widetilde{Z}_{[m]}$ i.i.d. copies with generic distribution that of $\widetilde{Z}_{[m]}$. Show that $\Pr \left\{ \widetilde{Z}_{[m]} = T_{\delta}^{(n)} \right\} \approx 2^{-nI(Z_B; Z_C|Z_A) + \pm 7n\delta}$.

3. (Conditional frequency typicality) For every $\delta > 0$, the following hold for all sufficiently large $n$. Prove them.

   a) $z_A^1 \in T_{\delta}^{(n)}(Z_A) \implies \Pr \left\{ Z_n^1 \in T_{\delta}^{(n)}(Z_A^1|z_A^1) \left| Z_A^1 = z_A^1 \right. \right\} \geq 1 - \delta$, so that for any $B \subseteq A^c$, $\Pr \left\{ Z_B^1 \in T_{\delta}^{(n)}(Z_B|z_A^1) \left| Z_A^1 = z_A^1 \right. \right\} \geq 1 - \delta$.

   b) $z_A^1 \in T_{\delta}^{(n)}(Z_A)$ and $B \subseteq A^c$, $\implies (1 - \delta)2^{nH(Z_B|Z_A) - 2n\delta} \leq \left| T_{\delta}^{(n)}(Z_B|z_A^1) \right| \leq 2^{nH(Z_B|Z_A) + 2n\delta}$.

4. Consider $(A^v, B^v)$. Given $A_i$, the random variable $B_i$ is independent of all other variables, for each $i = 1, 2, \cdots, n$. Prove that

$$I(A^n; B^n) \leq \sum_{i=1}^{n} I(A_i; B_i)$$

with equality if and only if $B_1, B_2, \cdots, B_n$ are independent.

Homework 1-1
5. Problem 15.6 (page 598) of Cover and Thomas (2nd edition).