We begin with some examples.

1 Examples

Example 1 [GMAC] Consider a Gaussian multiple-access channel (GMAC):

\[ Y = \sum_{k=1}^{m} X_k + Z. \]  

- Each user has an average power constraint \( P \). The average is over codewords and time. \( Z \sim N(0, \sigma^2) \)
- Users transmit independent data. So the power of \( \sum_{k=1}^{m} X_k \) is at most \( mP \). Under this constraint, even if they cooperate, \( \sum_{k \in S} R_k \leq C \left( \frac{|S|P}{\sigma^2} \right) \), \( \forall S \subseteq \{1, 2, 3, \ldots, m\} \), where \( C(x) = \frac{1}{2} \log(1 + x) \) is the Shannon capacity at SNR \( x \).
- For \( m = 2 \): Let \( P > 0 \).

\[
R_k \leq \frac{1}{2} \log \left( 1 + \frac{P}{\sigma^2} \right), \quad k = 1, 2
\]

\[
R_1 + R_2 \leq \frac{1}{2} \log \left( 1 + \frac{2P}{\sigma^2} \right)
\]

\[
= \frac{1}{2} \log \left( 1 + \frac{P}{\sigma^2} \right) + \frac{1}{2} \log \left( 1 + \frac{P}{P + \sigma^2} \right)
\]

\[
< 2 \cdot \frac{1}{2} \log \left( 1 + \frac{P}{\sigma^2} \right)
\]

So \( B'B \) will not allow both users to transmit at their respective capacities.

Figure 1: Capacity region (as we will see soon) of two user Gaussian MAC with transmit power constraint \( P \) and Gaussian channel noise power \( \sigma^2 \).
• Naive TDMA strategy: User 1 uses channel for \( \alpha \) fraction of the time. User 2 for \( 1 - \alpha \). Power constraint per (transmitted) sample remains \( P \). The achievable rate is \( \{\alpha C(\frac{P}{\sigma^2}), (1 - \alpha)C(\frac{P}{(1 - \alpha)\sigma^2})\} \).

![Figure 2: Naive TDMA: Achievable rate region of two user Gaussian multiple access channel with transmit power constraint \( P \) and Gaussian channel noise power \( \sigma^2 \).](image)

• Smart TDMA: User 1 can transmit at a higher power per sample given that it remains silent for a fraction of time. The average power constraints for both users, when user 1 gets the channel for a fraction \( \alpha \) of time is

\[
\left( \alpha \left( \frac{P}{\alpha}, 0 \right), (1 - \alpha) \left( 0, \frac{P}{1 - \alpha} \right) \right)
\]

so that \( \{(R_1, R_2) : R_1 = \alpha \frac{1}{2} \log(1 + \frac{P}{\alpha \sigma^2}) , R_2 = (1 - \alpha) \frac{1}{2} \log \left( 1 + \frac{P}{(1 - \alpha)\sigma^2} \right) \} \) is achievable.

![Figure 3: Smart TDMA: Achievable rate region of two user Gaussian multiple access channel with transmit power constraint \( P \) and Gaussian channel noise power \( \sigma^2 \).](image)

- Note: \( A, A' \) are achieved.
- In general,

\[
R_1 + R_2 = \alpha \frac{1}{2} \log(1 + \frac{P}{\alpha \sigma^2}) + (1 - \alpha) \frac{1}{2} \log \left( 1 + \frac{P}{(1 - \alpha)\sigma^2} \right) \\
\leq \frac{1}{2} \log \left( 1 + \frac{2P}{\sigma^2} \right), \text{ by Jensen’s inequality.}
\]

- By the condition for equality in Jensen’s inequality, observe that the point \( C \) given by \( (\frac{1}{4} \log(1 + \frac{2P}{\sigma^2}), \frac{1}{4} \log(1 + \frac{2P}{\sigma^2})) \) is achieved. \( C \) is a point on the outer bound’s boundary.

Lecture 1: Multiple Access Channels (MAC)-2
Example 2 [Multiplication Channel]

- \( Y = X_1 X_2, \quad X_k \in \{0, 1\}, \ k = 1, 2. \)
- In this example, there is no noise; multiple access interference (MAI) is the only source of information corruption.
- The extreme point \( A = (1, 0) \) (or \( A' = (0, 1) \)) can be achieved if user 2 (or user 1) transmits all 1s.
- By time-sharing, any point on the line \( AA' \) can be achieved.
- Since \( Y \) provides at most one bit of information, we expect \( R_1 + R_2 \leq 1 \). So the triangle \( OAA' \) is indeed the capacity region.

Example 3 [Addition Channel]

- \( Y = X_1 + X_2, \quad X_k \in \{0, 1\}, \ k = 1, 2 \), where the addition is integer addition.
- The extreme point \( A \) (or \( A' \)) can be achieved if user 2 (or user 1) transmits a deterministic sequence.
- No ambiguity if \( X_1 = X_2 = 0 \) or \( X_1 = X_2 = 1 \).
- Suppose user 1 sends 1 bit, \( X_1 \). User 2’s channel is then viewed as an erasure channel as shown in Figure 4. An erasure occurs to user 2 whenever both users send different bits. User 2 can thus send at most 1/2 bit. Receiver decodes user 2 first and then user 1 (point \( B \) in Figure 5). Similarly point \( B' \) can be achieved, and time-sharing gets us the line \( BB' \). From later results, this is indeed the capacity region.

\[ \begin{align*}
Y &= X_1 X_2, \quad X_k \in \{0, 1\}, \ k = 1, 2, \\
&\text{In this example, there is no noise; multiple access interference (MAI) is the only source of information corruption.}
&\text{The extreme point } A = (1, 0) \text{ (or } A' = (0, 1) \text{) can be achieved if user 2 (or user 1) transmits all 1s.}
&\text{By time-sharing, any point on the line } AA' \text{ can be achieved.}
&\text{Since } Y \text{ provides at most one bit of information, we expect } R_1 + R_2 \leq 1. \text{ So the triangle } OAA' \text{ is indeed the capacity region.}
\end{align*} \]

2 Definitions

**Definition 1 (DM–MAC)** A (two user) discrete memoryless multiple access channel (DM–MAC) denoted by \( (X_1, X_2, Y, p_{Y|X_1X_2}(y|x_1x_2)) \), consists of three finite sets \( X_1 \), \( X_2 \), and \( Y \) and a collection of probability mass functions \( p_{Y|X_1X_2}(y|x_1x_2) \) on \( Y \), one for each \( x_1x_2 \in X_1 \times X_2 \), with the interpretation that \( X_k \) is the input of user \( k \), \( k = 1, 2 \) and \( Y \) is the output. For \( n \in \mathbb{N} \), with \( X^n_k = (X_k, X_{k+1}, \ldots, X_{kn}) \), \( k = 1, 2 \) as inputs, the output sequence \( Y^n \) has pmf

\[
p_{Y^n|X^n_1X^n_2}(y^n|x^n_1x^n_2) = \prod_{i=1}^{n} p_{Y|X_1X_2}(y_i|x_{1i}x_{2i})
\]  \( \text{(2)} \)

Lecture 1 : Multiple Access Channels (MAC)-3
Definition 2 (Code) An \((n, M_1, M_2)\) code for the channel \((X_1, X_2, Y, p_{Y|X_1,X_2}(y|x_1,x_2))\) consists of the following:

1. An index set of messages for each user \(k\), \(\mathcal{W}_k = \{1, 2, \cdots, M_k\}\).
2. An encoder \(f_k\) for each user \(k\), \(f_k : \mathcal{W}_k \rightarrow \mathcal{X}_k^n\), \(k = 1, 2\). Note that \(\mathcal{W}_k \ni W_k \mapsto f_k(W_k) \in \mathcal{X}_k^n\). The codebook can be represented by an ordered set \(c = \{f_1(1), f_1(2), \cdots, f_1(M_1); f_2(1), f_2(2), \cdots, f_2(M_2)\}\).
3. A decoding rule, \(g : Y^n \rightarrow \phi \cup (\mathcal{W}_1 \times \mathcal{W}_2)\), i.e., \(y^n \mapsto g(y^n) = (\hat{w}_1, \hat{w}_2) \in \phi \cup (\mathcal{W}_1 \times \mathcal{W}_2)\). Note that \(g\) partitions \(Y^n\) into decision regions.

Definition 3 (Probability of error) Let \(W_k\) be the message transmitted by user \(k\) and let \(Y^n\) be the signal received. The conditional probability of error when \((W_1W_2) = (w_1w_2)\) was transmitted is given by
\[
P^{(n)}_{e_{w_1w_2}} (c) = Pr\{g(Y^n) \neq W_1W_2 | W_1W_2 = w_1w_2\}.
\]
The average probability of error for the code \(c\) is given by
\[
P_e^{(n)} (c) = \frac{1}{M_1M_2} \sum_{w_1w_2} P^{(n)}_{e_{w_1w_2}} (c).
\]
Note that the above equation assumes that all messages are equally likely and the users choose their messages independently.

Definition 4 (Achievability) The rate pair \((R_1, R_2)\) is achievable, if for every \(\eta > 0, \lambda \in (0, 1)\), there exists a sequence of \((n, M_1, M_2)\) codes that satisfy

1. \(P_e^{(n)} \leq \lambda\), and
2. \(\log_2 M_k \geq R_k - \eta\)

for all sufficiently large \(n\).

Definition 5 (Capacity region) The capacity region is the set of all achievable rate pairs, denoted by \(C_{\text{MAC}}\).

3 What can we expect?

- \(R_1 + R_2 \leq \max_{p(x_1,x_2)} I(X_1X_2;Y)\), (full cooperation)
- \(R_k \leq \max_{p(x_k)} \max_{x_{k^c}} I(X_k;Y|X_{k^c} = x_{k^c})\), \(k = 1, 2\) (the other user is benevolent)
4 Time Sharing

Lemma 6 \( \mathcal{C}_{\text{MAC}} \) is a closed convex set.

Proof: (convexity) The idea is time sharing. Let \( R = (R_1, R_2) \in \mathcal{C}_{\text{MAC}} \) and \( R' = (R'_1, R'_2) \in \mathcal{C}_{\text{MAC}} \). Fix \( t \in (0, 1) \). We will show \( t \mathcal{C} + (1 - t) \mathcal{C}' \subset \mathcal{C}_{\text{MAC}} \). For a given \( (\eta/2, \lambda/2) \), pick a sequence of \( (n, M_1, M_2) \) codes and another sequence of \( (n, M'_1, M'_2) \) codes such that for all sufficiently large \( n \),

\[
P_e^{(n)}(c) \leq \frac{\lambda}{2},
\]

\[
\frac{\log M_k}{n} > R_k - \frac{\eta}{2},
\]

\[
P_e^{(n)}(c') \leq \frac{\lambda}{2},
\]

\[
\frac{\log M'_k}{n} > R'_k - \frac{\eta}{2}.
\]

For each \( n \), use the code of length \( \lfloor tn \rfloor \) from the first sequence and the code of length \( n - \lfloor tn \rfloor \) from the second sequence. The overall probability of error, \( P_e^{(n)} \) is upper bounded by the sum of the individual codes’ errors. Since both \( \lfloor tn \rfloor \) and \( n - \lfloor tn \rfloor \to \infty \), we have for all sufficiently large \( n \),

\[
P_e^{(n)} \leq \frac{\lambda}{2} + \frac{\lambda}{2}.
\]

Since

\[
\log M_1 > \lfloor tn \rfloor R_1 - \eta/2
\]

\[
\log M_2 > \lfloor tn \rfloor R_2 - \eta/2
\]

\[
\log M'_1 > (n - \lfloor tn \rfloor)R'_1 - \eta/2
\]

\[
\log M'_2 > (n - \lfloor tn \rfloor)R'_2 - \eta/2.
\]

For all sufficiently large \( n \), the overall rate satisfies

\[
\frac{\log M_k M'_k}{n} = \frac{\log M_k}{n} + \frac{\log M'_k}{n} = \frac{\lfloor tn \rfloor \log M_k}{n} + \frac{n - \lfloor tn \rfloor \log M'_k}{n} \geq \frac{\lfloor tn \rfloor}{n} (R_k - \eta/2) + \frac{n - \lfloor tn \rfloor}{n} (R'_k - \eta/2)
\]

\[
= \frac{t}{n} (R_k - \eta/2) + (1 - t) (R'_k - \eta/2)
\]

\[
> \frac{t}{n} R_k + (1 - t) R'_k - \eta, \quad k = 1, 2.
\]

\[\blacksquare\]