\section{\( \mathcal{C} = \mathcal{D} \)}

\textbf{Theorem 1.} \( \mathcal{C} = \mathcal{D} \)

\textit{Proof.} (\( \mathcal{D} \subseteq \mathcal{C} \))

It is sufficient to show \( \mathcal{D}_1 := \text{conv} \left( \bigcup_{Z \in \mathcal{D}} \mathcal{C}(Z) \right) \subseteq \mathcal{C}_1 := \left( \bigcup_{Z \in \mathcal{D}} \mathcal{C}(Z) \right) \)

Let \( R \in \mathcal{D}_1 \). This implies \( \exists L \in \mathbb{N}, \exists Z^{(\ell)} \in \mathcal{D}, \ell \in [L], \exists R^{(\ell)} \in \mathcal{C}(Z^{(\ell)}), \ell \in [L], \) and \( \exists \lambda_\ell \geq 0, \ell \in [L], \) such that \( \sum_{\ell \in [L]} \lambda_\ell = 1 \) and

\[ R = \sum_{\ell \in [L]} \lambda_\ell R^{(\ell)}. \]

Since,

\[ R^{(\ell)} \in \mathcal{C}(Z^{(\ell)}), \quad \ell \in [L], \]

we have

\[ \sum_{\ell \in S} R^{(\ell)} \leq I(X^{\ell}_S; Y^{(\ell)} | X^{\ell}_{S^c}), \quad \ell \in [L] \]

and therefore,

\[ \sum_{\ell \in [L]} \lambda_\ell R^{(\ell)} \leq \sum_{\ell \in [L]} \lambda_\ell I(X^{(\ell)}_S; Y^{(\ell)} | X^{(\ell)}_{S^c}), \ell \in [L] = I(X_S; Y | X_{S^c}, Q), \]

for a suitably defined \( Z = QX_1X_2Y \in \mathcal{D}^*. \) Thus \( R \in \mathcal{C}(Z) \) for some \( Z \in \mathcal{D}^* \) and therefore \( R \in \mathcal{C}_1 \).

\textbf{We now prove the other part:} \( \mathcal{C} \subseteq \mathcal{D} \). Once again, it is sufficient to show that \( \mathcal{D}_1 \subseteq \mathcal{D} \).

Let \( R \in \mathcal{D}_1 \), i.e., \( R \in \mathcal{C}(Z) \) for some \( Z \in \mathcal{D}^* \).

\( \mathcal{C}(Z) \) is a polyhedron associated with a polymatroid.

By Edmonds’ result, \( R \) is dominated by a convex combination of the maximal extreme points of \( \mathcal{C}(Z) \).

We show that every maximal extreme point of \( \mathcal{C}(Z) \) is in \( \mathcal{D}_1 \) to complete the proof that \( R \in \mathcal{D} \).

Let \( r \in \mathcal{C}(Z) \) be a maximal extreme point. By Edmonds’ result, refer to fact in Lec. 5, \( r \) is a v(\( \pi \)) for some permutation \( \pi \), i.e.,

\[ r_{k_i} = \rho(\{k_1, k_2, \cdots, k_i\}) - \rho(\{k_1, k_2, \cdots, k_{i-1}\}), \quad i = 1, 2, \cdots, K \]

where \( k_1, k_2, \cdots, k_K \) is some permutation of \( [K] \). Expanding \( r_{k_i} \), we get

\[ r_{k_i} = I(X_{k_1}, X_{k_2}, \cdots, X_{k_i}; Y | X_{k_{i+1}}, \cdots, X_{k_K}, Q) - I(X_{k_1}, X_{k_2}, \cdots, X_{k_{i-1}}; Y | X_{k_1}, X_{k_{i+1}}, \cdots, X_{k_K}, Q) \]

\[ = \sum_{\ell=1}^{[Q]} p_\ell(S) \left[ \rho_\ell \left( \{k_1, k_2, \cdots, k_i\} \right) - \rho_\ell \left( \{k_1, k_2, \cdots, k_{i-1}\} \right) \right] \]

where \( \rho_\ell(S) = I(X_S; Y | X_{S^c}, Q = \ell) \). This implies that \( r \) is a convex combination of maximal extreme points of the polymatroidal polyhedra \( \mathcal{C}(Z^{(\ell)}) \), where \( Z^{(\ell)} \in \mathcal{D} \) and therefore \( r \in \mathcal{D}_1 \).

\( \square \)
2 Bounds on $|Q|$ 

Recall that $\mathcal{C} = \text{closure} \left( \bigcup_{Z \in \mathcal{P}^*} \mathcal{C}(Z) \right)$, where $Z = QX_1X_2Y$.

**Theorem 2.** (Caratheodory) If $A \subseteq \mathbb{R}^d$ and $a^* \in \text{conv } A$, then $a^* = \sum_{\ell=0}^{d} \lambda_\ell a^{(\ell)}$, where $a^{(\ell)} \in A$ and $\sum_{\ell=0}^{d} \lambda_\ell = 1, \lambda_\ell \geq 0, \forall \ell \in [d]$.

*Proof. Exercise. See Grunbaum for an elegant proof.*

**Theorem 3.** $\mathcal{C}$ does not reduce if we restrict $Z = QX_1X_2Y \in \mathcal{P}^*$ to those vectors such that $|Q| = 4$.

*Proof. Consider $Z = QX_1X_2Y \in \mathcal{P}^*$ with $Q$ taking values in $Q = \{1, 2, \ldots, |Q|\}$.

- Observe that $X_1^{(\ell)}X_2^{(\ell)}Y^{(\ell)} \sim p_{X_1X_2Y|Q}(\cdot|Q = \ell) \in \mathcal{P}$.
- Also, if $\overline{Q} \subseteq Q$, $\overline{Q}$ any random variable taking values in $\overline{Q}$, then $Z = \overline{Q}X_1X_2Y$ defined by $p_Z = p_{\overline{Q}X_1X_2Y}(\ell_{x_1x_2y}) = p_{\overline{Q}}(\ell)p_{X_1X_2Y|Q}(\cdot|Q = \ell) \in \mathcal{P}^*$.
- $\mathcal{C}(Z)$ is completely defined by

$$a = \begin{bmatrix} I(X_1;Y|X_2Q) \\ I(X_2;Y|X_1Q) \\ I(X_1X_2;Y|Q) \end{bmatrix} \in \text{conv } A,$$

where

$$A = \left\{ a^{(\ell)} = \begin{bmatrix} I(X_1;Y|X_2,Q = \ell) \\ I(X_2;Y|X_1,Q = \ell) \\ I(X_1X_2;Y|Q = \ell) \end{bmatrix}, \ \ell = 1, 2, \ldots, |Q| \right\} \subseteq \mathbb{R}^3.$$

- By Caratheodory's theorem, $\exists \overline{Q} = \{\ell_0, \ell_1, \ell_2, \ell_3\} \subseteq Q$ such that $a = \sum_{m=0}^{3} \lambda_{\ell_m} a^{(\ell_m)}$.
- Define $\overline{Q}$ as follows: $p_{\overline{Q}}(\ell_m) = \lambda_{\ell_m}, \ m = 0, 1, 2, 3$, to get $Z = \overline{Q}X_1X_2Y \in \mathcal{P}^*$.
- Easy to extend the above argument to $K$ users, in which case we need $|Q| = 2^K$.

\[ \square \]

Lecture 6 : Equivalence of $\mathcal{C}$ and $\mathcal{P}$, and Caratheodory’s theorem -2