1 Slepian–Wolf–Han Generalisation – Correlated Sources

Define \( \partial_k^- := \{ j \in [J] : \exists a \text{ connection between user } k \text{ and message source } j \} \).

Connections are specified via \( \partial_k^- \subseteq [J], k \in [K] \). Alternatively, they are via \( \partial_j^+ \subseteq [K], j \in [J] \). Let us call this system \( \text{MAC}(J, K, 1) \) with dependence on \( \partial^- \) and \( p_{Y|X[K]} \) suppressed.

\[ \mathcal{P} := \left\{ Z = U_{[J]} X_{[K]} Y \right\} \]

(1) \( U_j \) takes values in an arbitrary finite set \( U_j, j \in [J] \);
(2) \( X_k \) takes values in the given finite set \( X_k, k \in [K] \);
(3) \( Y \) takes values in the given finite set \( Y \);
(4) \( \exists f_k : \bigcup_{j \in \partial_k^-} U_j \to X_k, k \in [K] \),

\[ \text{satisfying } p_{X_k|U_{\partial_k^-}}(x_k|u_{\partial_k^-}) = p_{X_k}|U_{\partial_k^-} (x_k|u_{\partial_k^-}) = 1 \{ x_k = f_k(u_{\partial_k^-}) \} \]

(5) \[ p_{U_{[J]} X_{[K]} Y} = \left( \prod_{j \in [J]} p_{U_j} \right) \left( \prod_{k \in [K]} p_{X_k|U_{\partial_k^-}} \right) (p_{Y|X_{[K]}}). \]

Remark 1. \( U_{[J]} \to X_{[K]} \to Y \).

\[ \mathcal{P}^* := \left\{ Z = Q U_{[J]} X_{[K]} Y \right\} \]

(0) \( Q \) takes values in an arbitrary finite set \( Q \);
(1) – (4) as above, with \( f_k \) replaced by \( f_k(U_{\partial_k^-}|Q) \);
(5) \[ p_{Q U_{[J]} X_{[K]} Y} = p_Q \left( \prod_{j \in [J]} p_{U_j|Q} \right) \left( \prod_{k \in [K]} p_{X_k|U_{\partial_k^-} Q} \right) (p_{Y|X_{[K]}}). \]
• For $S \subseteq J$, define
\[ \rho(S) := \begin{cases} 
0, & \text{if } S = \emptyset \\
I(X_S; Y_{X_S}) & \text{otherwise}
\end{cases} \]

Note that $\rho(\cdot)$ is a submodular rank function.

• For $Z \in \mathcal{P}^*$, $\mathcal{C}(Z)$ is the polyhedron associated with the polymatroid $([J], \rho)$.

• Define $\mathcal{C}$ and $\mathcal{D}$ as before.

**Theorem 1.** $\mathcal{C}_{\text{MAC}}(J, K, 1) = \mathcal{C} = \mathcal{D}$.

**Proof.** Read Han.

No new elements except for a careful application of Fano’s inequality, for each subset $S \subseteq [J]$. \qed

**Remark 2.** Cardinality bounds for $Q$, and $\cup_j, j \in [J]$ require a generalization of Caratheodory’s theorem. Read Han’s paper for cardinality bounds on $\cup_j$ when $|Q| = 1$.

## 2 Ahlswede–Ulrey–Han Generalisation – Multiple Output Terminals

![Figure 2: Multiple Access Channel with Correlated Sources and Multiple Output Terminals.](image)

- Every output terminal is interested in all source symbols.

- One may view this as a compound channel. Two formulations possible: (1) $P_e^{(n)}(\ell) \leq \lambda, \forall \ell \in [L]$; (2) $P_e^{(n)} := \Pr\{g_\ell(Y^n_\ell) \neq W_{[J]}, \text{ for some } \ell \in [L]\} \leq \lambda$.

**Remark 3.** (A simplification) The capacity of MAC($J, K, L$) with a general $p_{Y|X_{[K]}}$ is the same as the capacity of a MAC($J, K, L$) with
\[ P_{Y_{[L]|X_{[K]}}} = \prod_{\ell \in [L]} p_{Y_{\ell}|X_{[K]}}. \]

This is because, given $Y_\ell$, $P_e^{(n)}(\ell)$ does not depend on $Y_{[L]\setminus\ell}$.

**Remark 4.** The channel capacity remains the same under the two formulations (because $\max_{\ell \in [L]} P_e^{(n)}(\ell) \leq P_e^{(n)} \leq \sum_{\ell \in [L]} p_e^{(n)}(\ell)$). So, we may restrict our attention to conditionally independent outputs.
We can therefore set $P := \begin{cases} Z = QU_{[J]}X_{[K]}Y \text{ such that (0)-(5) below hold} \end{cases}$:

1. $Q$ takes values in an arbitrary finite set $Q$;
2. (0) - (4) holds as above, where in (4) $f_k$ is replaced by $f_k(U_{[J]}Q)$;
3. $\mathcal{C}(Z)$ is the polygon associated with $([K], \rho)$ where
   $$\rho(S) := \min_{\ell\in[L]} \rho_\ell(S)$$
   and
   $$\rho_\ell(S) := I(U_S; Y_\ell|U_S, Q)$$

**Remark 5.** $\rho$ is not necessarily a rank function. See Fig. 3 below.

**Theorem 2.** (Han, 79) $C_{MAC}(J, K, L) = \text{closure} \left( \bigcup_{Z\in\mathcal{P}^*} \mathcal{C}(Z) \right) = \mathcal{C}$.

**Proof.** Typicality at each terminal is with respect to $T_\delta^{(n)}(QU_{[J]}X_{[K]}Y_{[L]})$.

**Remark 6.** Cardinality bounds apply.

**Note 1.** $\mathcal{P} \subsetneq \mathcal{C}$. There is an error in Thm. 5.1 of Han. See Remark 2.2 of Han-Kobayashi p.53.

**Example 1 (Gaussian).** $Y = X_k + W_k$.

- $\bigcup_{Z\in\mathcal{P}^*} \mathcal{C}(Z) = \mathcal{C}(Z_G)$, where $Z_G = X_{[K],G}Y$, $X_{[K],G} \sim N(0, \text{diag}(P_1, P_2, \ldots, P_K))$, so that
- $\mathcal{C}_{MAC}(K, K, 1) = \left\{ R \in \mathbb{R}^+: R(S) \leq \frac{1}{2} \log \left( 1 + \frac{P(S)}{\sigma^2} \right), \forall S \subseteq [K] \right\}$.

**Example 2 (Gaussian vector MAC with noise covariance $\sum$).**

$$\mathcal{C}_{MAC}(K, K, 1) = \left\{ R \in \mathbb{R}^+: R(S) \leq \frac{1}{2} \log \left| I_K + \sum_{S \subseteq [K]}^{-1} sP(S)s^T \right|, \forall S \subseteq [K] \right\}$$

where $s = [s_1, s_2, \ldots, s_K], s_k \in \mathbb{R}^{N_k}$, signature of user $k$. 

Lecture 7 : Han’s generalisations-3