1 Definitions

Definition 1 (DM-IC). A (two user) discrete memoryless interference channel (DM-IC) denoted by $(\mathcal{X}_1, \mathcal{X}_2, \mathcal{Y}_1, \mathcal{Y}_2, p_{Y_1Y_2|X_1X_2}(y_1y_2|x_1x_2))$, consists of finite sets $\mathcal{X}_1, \mathcal{X}_2, \mathcal{Y}_1$, and $\mathcal{Y}_2$, and a collection of probability mass functions $p_{Y_1Y_2|X_1X_2}(y_1y_2|x_1x_2)$ on $\mathcal{Y}_1 \times \mathcal{Y}_2$, one for each $x_1x_2 \in \mathcal{X}_1 \times \mathcal{X}_2$, with the interpretation that $X_k$ is the input of user $k$, $k = 1, 2$ and $Y_k$ is the input to the decoder of user $k$. For $n \in \mathbb{N}$, with $X^n_k = (X_{k1}, X_{k2}, \ldots, X_{kn})$, $k = 1, 2$ as inputs, the output sequence $Y^n_1Y^n_2$ has pmf

$$p_{Y^n_1Y^n_2|X^n_1X^n_2}(y^n_1y^n_2|x^n_1x^n_2) = \prod_{i=1}^n p_{Y_1Y_2|X_1X_2}(y_1y_2|x_1x_2) \quad (1)$$

Definition 2 (Code). An $(n, M_1, M_2)$ code for the channel $(\mathcal{X}_1, \mathcal{X}_2, \mathcal{Y}_1, \mathcal{Y}_2, p_{Y_1Y_2|X_1X_2}(y_1y_2|x_1x_2))$ consists of the following:

1. An index set of messages for each user $k$, $\mathcal{W}_k = \{1, 2, \ldots, M_k\} = [M_k]$.
2. An encoder $f_k$ for each user $k$, $f_k : [M_k] \rightarrow \mathcal{X}^n_k$, $k = 1, 2$. Note that $[M_k] \ni W_k \mapsto f_k(W_k) \in \mathcal{X}^n_k$.
   The codebook can be represented by an ordered set
   $$c = \{f_1(1), f_1(2), \ldots, f_1(M_1); f_2(1), f_2(2), \ldots, f_2(M_2)\}.$$  
3. A decoding rule, $g_k : \mathcal{Y}^n_k \rightarrow \phi \cup [M_k], k = 1, 2$, i.e., $y^n_k \mapsto g_k(y^n_k) = \hat{w}_k \in \phi \cup [M_k]$. Note that $g_k$ partitions $\mathcal{Y}^n_k$ into decision regions.

Definition 3 (Probability of error). Let $W_k$ be the message transmitted by user $k$ and let $Y^n_k$ be the signal received by user $k$. The conditional probability of error for user $k$ when $(W_1W_2) = (w_1w_2)$ was transmitted is given by

$$P^{(n)}_{c,w_1w_2}(c;k) = Pr\{g_k(Y^n_k) \neq W_k|W_1W_2 = w_1w_2\}, \quad k = 1, 2.$$

The average probability of error for the code $c$ (for user $k$) is given by

$$P^{(n)}_c(c;k) = \frac{1}{M_1M_2} \sum_{w_1w_2} P^{(n)}_{c,w_1w_2}(c;k) \quad k = 1, 2.$$

Note that the above equation assumes that all messages are equally likely and the users choose their messages independently.
Definition 4 (Achievability). The rate pair \((R_1, R_2)\) is achievable, if for every \(\eta > 0, \lambda \in (0, 1)\), there exists a sequence of \((n, M_1, M_2)\) codes that satisfy

1. \(P_e^{(n)}(k) \leq \lambda\), and
2. \(\frac{\log M_k}{n} > R_k - \eta, k = 1, 2\)

for all sufficiently large \(n\).

Definition 5 (Capacity region). The capacity region is the set of all achievable rate pairs, denoted by \(\mathcal{C}_I\).

Remark 1.

- \(\mathcal{C}_I\) is closed and convex.
- \(R \in \mathcal{C}_I, R\) dominates \(r \implies r \in \mathcal{C}_I\).
- \(\mathcal{C}_I\) depends on \(p_{Y_1,Y_2|X_1,X_2}\) only through the marginals \(p_{Y_k|X_1,X_2}, k = 1, 2\).

Example 1. (Gaussian interference channel. Caution: \(X_k = \mathbb{R}\))

\[
\begin{bmatrix}
Y_1 \\
Y_2
\end{bmatrix} = \begin{bmatrix}
\frac{1}{\sqrt{a_{12}}} & \sqrt{a_{12}} \\
\sqrt{a_{21}} & 1
\end{bmatrix} \begin{bmatrix}
X_1 \\
X_2
\end{bmatrix} + \begin{bmatrix}
\xi_1 \\
\xi_2
\end{bmatrix}
\]

where \(\xi_k \sim N(0,1)\); \(X_1\) and \(X_2\) are power constrained

\[
\frac{1}{n} \sum_{i=1}^{n} x_k^2(w_k) \leq P_k, \ \forall w_k \in [M_k].
\]

- \(\mathcal{C}_I\) for this channel is not fully known. But,

Proposition 1. For the Gaussian interference channel, if \(a_{12} \gg 1\) and \(a_{21} \gg 1\), then \(\mathcal{C}_I = \mathcal{C}_{MAC(2,2,2)}\) is given by

\[
\mathcal{C}_I = \left\{ R \in \mathbb{R}^2_+ : R_k \leq \frac{1}{2} \log (1 + P_k), k = 1, 2, \\
R_1 + R_2 \leq \min \left\{ \frac{1}{2} \log (1 + P_1 + a_{12}P_2), \frac{1}{2} \log (1 + P_2 + a_{21}P_1) \right\} \right\}
\]

Remark 2.

- \(\mathcal{C}_I\) is not necessarily a polyhedron associated with a polymatroid.
- \(a_{12} \geq 1 + P_1\) and \(a_{21} \geq 1 + P_2 \implies \mathcal{C}_I\) is a rectangle; as if there were no interference.

Proof. Achievability: Make both decoders decode both streams. Hence, \(\mathcal{C}_{MAC(2,2,2)} \subseteq \mathcal{C}_I\).

Converse: \(\Pi a_{ij} \geq 1, i \neq j\), then the other user sees a stronger channel and should be able to decode the unwanted signal. Let \(R \in \mathcal{C}_I\). For a fixed \(\lambda/2 \in (0,1), \eta > 0\), consider a sequence of \((n,M_1,M_2)\) codes that satisfy \(P_e^{(n)}(k) \leq \lambda/2\) and the rate condition.

In vector notation, the received signals are

\[
y_1^n = x_1^n + \sqrt{a_{12}} x_2^n + \xi_1^n
\]

\[
y_2^n = \sqrt{a_{21}} x_1^n + x_2^n + \xi_2^n.
\]

Also, \(f_k(g_k(y_k^n)) = x_k^n\) with probability \(\geq 1 - \lambda/2\). Consider,

\[
g_k^n = \frac{y_2^n - f_2\left(\frac{g_2(y_2^n)}{\sqrt{a_{21}}}\right)}{\sqrt{a_{21}}} + \sqrt{a_{12}} f_2\left(\frac{g_2(y_2^n)}{\sqrt{a_{21}}}\right) + \sqrt{1 - \frac{1}{a_{21}}} \xi_1^n.
\]

Lecture 8 : Interference Channels-2
where $\tilde{\xi}_1^n$ is independent Gaussian noise generated at the receiver. Similarly obtain $\tilde{y}_2^n$. Define $G_1(y_1^n) := (g_1(y_1^n), g_2(y_2^n))$. Similarly define $G_2$. Define:

$$
\begin{bmatrix}
\tilde{Y}_1^n \\
\tilde{Y}_2^n
\end{bmatrix} := \begin{bmatrix}
\frac{1}{\sqrt{a_{21}}} & \sqrt{a_{12}} \\
-\frac{1}{\sqrt{a_{21}}} & 1
\end{bmatrix}
\begin{bmatrix}
x_1^n \\
x_2^n
\end{bmatrix} + \begin{bmatrix}
\tilde{\xi}_1^n \\
\tilde{\xi}_2^n
\end{bmatrix},
$$

where $\tilde{\xi}_1^n = \frac{\xi_1^n}{\sqrt{a_{21}}} + \sqrt{1 - \frac{1}{a_{21}}} \xi_1^n$ and $\tilde{\xi}_2^n = \frac{\xi_2^n}{\sqrt{a_{12}}} + \sqrt{1 - \frac{1}{a_{12}}} \xi_2^n$.

Since, $\begin{bmatrix}
\tilde{Y}_1^n \\
\tilde{Y}_2^n
\end{bmatrix}$ has the same marginal as $\begin{bmatrix}
Y_1^n \\
Y_2^n
\end{bmatrix}$ given $\begin{bmatrix}
X_1^n \\
X_2^n
\end{bmatrix}$, we must have

$$
\Pr\{g_1(\tilde{Y}_1^n) \neq W_1|W_1W_2 = w_1w_2\} = P_e^{(n)}_{\psi,w_1w_2}(1) \quad \text{and} \quad \Pr\{g_1(\tilde{Y}_1^n) \neq W_1|W_1W_2 = w_1w_2\} = P_e^{(n)}_{\psi,w_1w_2}(2).
$$

Therefore,

$$
\Pr\{G_2(Y_2^n) \neq W_1|W_1W_2 = w_1w_2\} \leq \Pr\{\tilde{Y}_2^n \neq Y_2^n|W_1W_2 = w_1w_2\} + \Pr\{g_1(\tilde{Y}_1^n) \neq W_1|W_1W_2 = w_1w_2\} \leq P_e^{(n)}_{\psi,w_1w_2}(1) + P_e^{(n)}_{\psi,w_1w_2}(2).
$$

Similarly, $\Pr\{G_2(Y_2^n) \neq W_1|W_1W_2 = w_1w_2\} \leq P_e^{(n)}_{\psi,w_1w_2}(1) + P_e^{(n)}_{\psi,w_1w_2}(2)$. Averaging over messages, we have a joint decoder whose $P_e^{(n)}_{\psi,\text{joint}}(k) \leq \lambda/2 + \lambda/2 = \lambda$. So $R \in C_{\text{MAC}(2,2,2)}$. □

## 2 A Modified Interference Channel

**Motivation:** While in the above example, $a_{12} \geq 1$, $a_{21} \geq 1$ implies both decoders could decode both streams, this is not possible in general. Suppose each decoder decodes only a part of the other user’s signal. Let us call this common. The undecoded signal is dedicated to the other user. The part that is decoded may enable a partial interference cancellation/mitigation. This motivates a splitting of the users data into a common and a dedicated portion.

![Modified Interference Channel](image)

**Definition 6 (Code).** An $(n, M_1, M_2, M_3, M_4)$ code for the channel $(X_1, X_2, Y_1, Y_2, p_{Y_1|Y_2,X_1,Y_2}(y_1|y_2,x_1,x_2))$ consists of the following:

1. $[M_1] \times [M_2]$ is the index set of messages for user 1 and $[M_3] \times [M_4]$ that for user 2.

2. An encoder for user 1, $f_1 : [M_1] \times [M_2] \to X_1^n$ and that for user 2 is $f_2 : [M_3] \times [M_4] \to X_2^n$. The codebook can be represented by an ordered set $c = \{f_1(\{1,1\}), f_1(\{1,2\}), \ldots, f_1(\{1,M_2\}), \ldots, f_1(\{M_1,M_2\}); f_2(\{1,1\}), \ldots, f_2(\{M_3,M_4\})\}$.  

3. A decoding rule, $g_1 : Y_1^n \to \phi \cup [M_3] \times [M_4]$, and $g_2 : Y_2^n \to \phi \cup [M_2] \times [M_3] \times [M_4]$.

**Definition 7 (Probability of error).** Let $(W_1, W_2)$ be the message transmitted by user 1 and $(W_3, W_4)$ by user 2. $P_e^{(n)}_{\psi,w_1w_2w_3w_4}(\ell)$, the conditional probability of error for output terminal $\ell$ when $(W_1W_2W_3W_4) = (w_1w_2w_3w_4)$ was transmitted is defined as before. and $P_e^{(n)}(\ell)$, the average probability of error for the code $c$ (for output terminal $\ell$) is also defined as before.

Lecture 8 : Interference Channels-3
Definition 8 (Achievability). The rate quadruple \((R_1, R_2, R_3, R_4)\) is achievable, if for every \(\eta > 0\), \(\lambda \in (0, 1)\), there exists a sequence of \((n, M_1, M_2, M_3, M_4)\) codes that satisfy

1. \(P_e^n(\ell) \leq \lambda\), and
2. \(\frac{\log M_\ell}{n} > R_\ell - \eta\), \(\ell \in [2]\),

for all sufficiently large \(n\).