1 Fourier Motzkin elimination:

Solve \( x_1, x_2, \ldots, x_n \) such that
\[
\sum_{j=1}^{n} a_{ij} x_j \leq b_i, \quad i = 1, 2, \ldots, m.
\]

Pick a variable, say \( x_n \). Eliminate it as follows. Assume \( a_{in} \neq 0 \).
\[
a_{in} x_n \leq b_i - \sum_{j=1}^{n-1} a_{ij} x_j, \quad i = 1, 2, \ldots, m.
\]

If \( a_{in} > 0 \), then upper bound \( x_n \leq \frac{b_i}{a_{in}} - \sum_{j=1}^{n-1} \frac{a_{ij}}{a_{in}} x_j = \beta_i \), otherwise lower bound \( x_n \geq a_i \). Eliminate \( x_n \) from all equations; replace by \( \alpha_i \) \( \leq \beta_i \) for every \( i \) such that \( i \) yields a lower bound, \( i \) yields an upper bound. Let \( LB_n \) and \( UB_n \) be the set of indices that yield lower and upper bounds respectively.

If this system has a solution in \( n-1 \) variables, then that solution with any \( x_n \) in \( \left[ \max_{i \in LB_n} \min_{i \in UB_n} \beta_i \right] \) is a solution to the original set.

2 Gaussian interference channel:

For the Gaussian interference channel defined earlier, \( \mathcal{P}, \mathcal{P}^* \) depend on the power constraints \( P_1 \) and \( P_2 \). Note the following definitions.

- \( \mathcal{P}^*(P_1, P_2) = \{ Z \in \mathcal{P}^*, \operatorname{Var}(X_k) \leq P_k, k = 1, 2 \} \)
- \( \mathcal{P}(P_1, P_2) = \{ Z \in \mathcal{P}^*(P_1, P_2), \text{with } |Q| = 1 \} \)
- \( \mathcal{G} = \text{closure conv } \bigcup_{Z \in \mathcal{P}(P_1, P_2)} \mathcal{A}(Z) \)
- \( \mathcal{G}^* = \text{closure } \bigcup_{Z \in \mathcal{P}^*(P_1, P_2)} \mathcal{A}(Z) \)
- \( \mathcal{P}'(P_1, P_2) = \{ Z \in \mathcal{P}(P_1, P_2) : U_1, U_2, U_3, U_4 \text{ are Gaussian, } U_1 + U_2 = X_1, U_3 + U_4 = X_2 \} \)
- \( \mathcal{G}' = \text{closure conv } \bigcup_{Z \in \mathcal{P}'(P_1, P_2)} \mathcal{A}(Z) \).

Questions: Does correlation in \( U_1U_2U_3U_4 \) help? Is \( \mathcal{G}' \subseteq \mathcal{G}^* \)? Is \( \mathcal{G}^* \subseteq \mathcal{G}' \)?

3 Outer bounds:

(1) DMC.

Definition 1 (II).
\[
\mathcal{Q} := \{ Z = QX_1X_2Y_1Y_2 \tilde{Y}_1 \tilde{Y}_2 \text{ such that (1) – (2) below hold} \}.
\]
Figure 1: A statistical model for outer bound.

(1) \[ p_Z = pq \left( p_{X_1 | Q} p_{X_2 | Q} \right) \left( p_{Y_1 Y_2 | X_1 X_2} \right) \left( p_{Y_1 Y_2 | X_1 X_2 Y_1} \right) \left( p_{Y_1 | X_2 Y_2} \right) \]

(2) \[ p_Y | X_1 X_2 = p_Y | X_1 X_2, p_{Y_1} | X_1 = p_{Y_1} | X_1 X_2. \]

Definition 2.
\[
\mathcal{R}_\Pi (Z) := \left\{ (R_1, R_2) \in \mathbb{R}_+^2 : \begin{array}{l}
R_1 \leq I(X_1; Y_1 | X_2 Q) \\
R_2 \leq I(X_2; Y_2 | X_1 Q) \\
R_1 + R_2 \leq \min \left\{ I \left( X_1 X_2; Y_1 \bar{Y}_1 | Q \right), I \left( X_1 X_2; Y_2 \bar{Y}_2 | Q \right) \right\} \end{array} \right\}.
\]

Definition 3.
\[
\mathcal{R}_\Pi := \text{closure} \bigcup_{Z \in \mathcal{R}_\Pi} \mathcal{R}_\Pi (Z)
\]

Theorem 1. \( \mathcal{C}_1 \subseteq \mathcal{R}_\Pi \)

Proof. (1) Fix \( n \), \( p_Q(i) = \frac{1}{n} p_{X_1 X_2 Y_1 Y_2 \mid Q}(x_1 x_2 y_1 y_2 \mid i) = p_{X_1 X_2 Y_1 Y_2}. \) As in the MAC’s converse, \( R_1 \leq I(X_1; Y_1 | X_2 Q) \) and \( R_2 \leq I(X_2; Y_2 | X_1 Q) \).

(2) • Now suppose the same codes are used in the new DMC with outputs \( Y_1 \bar{Y}_1 \) at decoder 1 and \( Y_2 \bar{Y}_2 \) at decoder 2.
• Decoder 1 gets \( X_1^n, \bar{Y}_1^n, Y_1^n \); sends \( \bar{Y}_1^n \) to DMC \( p_{Y_1 \mid X_1} \) to get \( \bar{Y}_2^n \), applies decoder 2’s decode function to get \( \hat{W}_2 \) as reliably as decoder 2.
• Analogously, decoder 2 gets \( \hat{W}_2 \) reliably, and moreover \( \hat{W}_1 \) as reliably as decoder 1.
• Using the converse to Ahlswede–Ulrey–Han generalisation, since both can decode, we have a compound DMC that satisfies
\[
R_1 + R_2 \leq I(X_1 X_2; Y_1 \bar{Y}_1 | Q) \\
R_1 + R_2 \leq I(X_1 X_2; Y_2 \bar{Y}_2 | Q)
\]

Lecture 10: Fourier-Motzkin elimination, outer bounds of interference channels-2