A  Recall:

\[ \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{a_{12}} & \sqrt{a_{12}} \\ \sqrt{a_{12}} & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix}. \]

A.1 
\[ a_{12} \geq 1, a_{21} \geq 1 \implies \mathcal{C}_I = \left\{ \begin{array}{c} R_1 \leq \frac{1}{2} \log (1 + P_1) \\ R_2 \leq \frac{1}{2} \log (1 + P_2) \\ R_1 + R_2 \leq \min \left\{ \frac{1}{2} \log (1 + P_1 + a_{12}P_2), \frac{1}{2} \log (1 + P_2 + a_{21}P_1) \right\} \end{array} \right\} \]

A.2 
\[ a_{12} \geq 1 + P_1 \text{ and } a_{21} \geq 1 + P_2 \implies \text{interference does not affect capacity region, } \mathcal{C}_I = [0, C(P_1)] \times [0, C(P_2)], \]
where \( C(P) := \frac{1}{2} \log (1 + P) \).

B  Degraded interference channel:

One receiver sees a statistically noisy version of the other’s received signal.

B.1
- Since, \( \mathcal{C}_I \) depends only on marginals, we may assume physical degradedness (more on this when we do broadcast channels).
- \( y_1 = x_1 + \sqrt{a_{12}} x_2 + \xi_1 \)
  \( y_2 = \sqrt{a_{21}} x_1 + x_2 + \xi_2 \).
  If \( y_2 \) is a degraded version of \( y_1 \), we must have \( \sqrt{a_{21}} x_1 + x_2 = c \cdot (x_1 + \sqrt{a_{12}} x_2) \) \( \forall x_1, x_2 \), with \( c \in (0, 1) \). Thus \( a_{21} = c \) and \( a_{12} a_{21} = 1 \).
- \( a_{21} = 1 \) is solved. So WMA \( a_{21} \in (0, 1) \).
- Equivalently, a degraded interference channel satisfies
  \[ \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & c^{-1} \\ c & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix}, \quad c \in (0, 1). \]

C  Z–channel:

\[ \begin{align*}
  y_1 &= x_1 + \xi_1 \\
  y_2 &= \sqrt{a_{21}} x_1 + x_2 + \xi_2.
\end{align*} \]

We will denote its capacity region \( \mathcal{C}_{I,Z}(a_{21}) \).
C.1

**Proposition 1.** (1) If \( a_{21} \geq 1 \), then
\[
\mathcal{C}_I = \left\{ \begin{array}{l}
R_1 \leq C(P_1) \\
R_2 \leq C(P_2) \\
R_1 + R_2 \leq C(P_2 + a_{21} P_1)
\end{array} \right\}.
\]

(2) If \( a_{21} \geq 1 + P_2 \), then \( \mathcal{C}_I = [0, C(P_1)] \times [0, C(P_2)]. \)

**Proof.** (1)

\[\text{ Figure 1: Achievable rate region when } 1 \leq a_{21} \leq 1 + P_2.\]

\begin{itemize}
\item \text{case 1: } \( C(P_1) < C(P_2 + a_{21} P_1) - C(P_2) \implies 1 + P_1 < \frac{1 + P_2 + a_{21} P_1}{1 + P_2} = 1 + \frac{a_{21} P_1}{1 + P_2} \implies a_{21} > 1 + P_2. \)
\end{itemize}

Claim: \( \mathcal{C}_I = [a, C(P_1)] \times [0, C(P_2)]. \)

**Proof.** Strategy to achieve \( (C(P_1), C(P_2)) \):

- User 1 encodes as in a single user channel with rate \( C(P_1) \). Decoder 1 can decode this signal.
- User 2 encodes in a similar fashion. Decoder 2 decodes signal 1 first. \( \text{SNR} = \frac{a_{21} P_1}{P_2 + 1} > P_1. \)
  \( \implies \) decoding is reliable. Subtract, decode user 2’s signal.

\begin{itemize}
\item \text{case 2: } 1 \leq a_{21} \leq 1 + P_2
\end{itemize}

Claim: \( \mathcal{C}_I \) is the shaded region.

**Proof.**

- \( A \) is achievable. User 1 encodes at \( C\left(\frac{a_{21} P_1}{1 + P_2}\right) \) as in single user channel. User 2 encodes at \( C(P_2) \).
- Decoder 1 sees \( P_1 \), but rate is lower (corresponding \( \text{SNR} \) is \( \frac{a_{21} P_1}{1 + P_2} \)). So decoder 1 decodes message 1. Decoder 2 employs successive interference cancellation.
- \( B \) is achievable. User 1 encodes at \( C(P_1) = C\left(\frac{1}{a_{21}}(a_{21} P_1)\right) \). User 2 splits powers as \( a_{21} - 1, P_2 - (a_{21} - 1) \geq 0 \) and encodes at rates \( C(a_{21} - 1), C\left(\frac{P_2 - (a_{21} - 1)}{1 + (a_{21} - 1) + a_{21} P_1}\right) \). Clearly decoder 1 can decode at \( C(P_1) \), its capacity. Decoder 2 decodes \( C\left(\frac{P_2 - (a_{21} - 1)}{1 + (a_{21} - 1) + a_{21} P_1}\right) \), then \( C(P_1) = C\left(\frac{1}{a_{21}}(a_{21} P_1)\right) \), then finally \( C(a_{21} - 1) \). Sum rate is \( C(1 + \frac{P_2 - a_{21}}{a_{21}(1 + P_1)}) + C(P_1) + C(a_{21} - 1) = C(P_2 + a_{21} P_1) \)

(2) Obvious. Since \( C(P_2 + a_{21} P_1) \geq C(P_1) + C(P_2). \)

Lecture 11: Gaussian interference channels, degraded interference channels, Z-channel-2
C.2

**Remark 1.** \( \mathcal{C}_I \), when \( a_{21} \in (0,1) \) is still open.

C.3

**Proposition 2.** The capacity region of the Z-channel with \( a_{21} \in (0,1) \) equals the capacity region of the degraded interference channel with the same \( a_{21} \).

**Proof.** (Transformation)

\[ C(a) \leq C(b) \leq C(c) \]

Also, \( C(d) \leq C(c) \). This is because \( Y_2 \) has the same conditional output distribution in both cases. To decoder 1, provide \( x_2^n \) and \( x_1^n + \xi_1^n \). Optimal use of \( x_2^n \) and \( x_1^n + \xi_1^n \) by decoder 1 bounds \( C(d) \). Clearly \( y^n_1 \) can be discarded. Moreover, for decoding \( w_1 \), \( x_2^n \) provides no additional information and can be discarded. Optimal decoding then makes use of \( x_1^n + \xi_1^n \) only, as in (c). So \( C(d) \leq C(c) \).

- \( C(a) = C(b) = C(c) \) is clear. Also, \( C(d) \subseteq C(c) \). This is because \( Y_2 \) has the same conditional output distribution in both cases. To decoder 1, provide \( x_2^n \) and \( x_1^n + \xi_1^n \). Optimal use of \( x_2^n \) and \( x_1^n + \xi_1^n \) by decoder 1 bounds \( C(d) \). Clearly \( y^n_1 \) can be discarded. Moreover, for decoding \( w_1 \), \( x_2^n \) provides no additional information and can be discarded. Optimal decoding then makes use of \( x_1^n + \xi_1^n \) only, as in (c). So \( C(d) \subseteq C(c) \).

- \( C(c) \subseteq C(d) \): Take a code on (c). We will apply it on (d). User 2’s decoder remains the same. Modify user 1’s decoder to apply user 2’s decoder first (after addition of noise), subtract user 2’s signal, apply user 1’s (c) decoder. We have thus simulated channel (c) on channel (d) with probability of decoder 2 error. so an achievable rate on (c) is achievable in (d).

\[ \square \]
D A bound on $\mathcal{C}_I$ for the degraded interference channel:

D.1 Proposition 3. a) Let $a_{21} \in (0,1), a_{12} = 1/a_{21}$, so that $y_2$ is a degraded version of $y_1$. Then,

$$\mathcal{C}_I \subseteq \left\{ \begin{array}{l}
R_1 \leq \frac{1}{2} \log \left( 1 + t(P_1 + P_2/a_{21}) \right) \\
R_2 \leq \frac{1}{2} \log \left( 1 + (1-t)\frac{P_1 + P_2/a_{21}}{1/a_{21} + t(P_1 + P_2/a_{21})} \right)
\end{array} \right\} =: \mathcal{C}_{BC} \left( P_1 + \frac{P_2}{a_{21}} \right)$$

$$\cap \ [0, C(P_1)] \times [0, C(P_2)]$$

b) Moreover, $\left( c(P_1), c \left( \frac{P_2}{1 + a_{21}P_1} \right) \right)$ is achievable.

c) $$\max_{(R_1, R_2) \in \mathcal{C}_I} R_1 + R_2 = C(P_1) + C \left( \frac{P_2}{1 + a_{21}P_1} \right).$$

Proof. a) Pool resources and consider the broadcast channel.

$$y_1 = x_1 + \frac{1}{\sqrt{a_{21}}} x_2 + \xi_1$$

$$y'_2 = \frac{1}{\sqrt{a_{21}}} y_2 = x_1 + \frac{1}{\sqrt{a_{21}}} x_2 + \frac{\xi_2}{\sqrt{a_{21}}},$$

$$\mathcal{C}_I \subseteq \mathcal{C}_{BC} \left( P_1 + \frac{P_2}{a_{21}} \right)$$

which is given by the first set in the upper bound (refer sec. E.2). Furthermore, $\mathcal{C}_I \subseteq [0, C(P_1)] \times [0, C(P_2)]$. This yields (a).

b) Choose $t$ so that $t(P_1 + P_2/a_{21}) = P_1$. Then $(1-t)(P_1 + P_2/a_{21}) = \frac{P_2}{a_{21}}$. Since superposition coding in the BC disregards the stronger signal knowledge, and power constraints at the two terminals are met, the encoding can happen at separate places with superposition done by the channel. So $\left( c(P_1), c \left( \frac{P_2}{1 + a_{21}P_1} \right) \right)$ is achievable.

c) Figure 3: Outer bound of $\mathcal{C}_I$. 

Lecture 11 : Gaussian interference channels, degraded interference channels, Z-channel-4
boundary parametrized by $t$ is concave function of $R_1$. |Slope| 1, negative. $\Rightarrow$ max $R_1 + R_2 \leq$ max value of $A_1$. But $A_1$ is achievable.

\[ \mathcal{C}_{BC} \]

\section{Putting our results together:}

\subsection{Proposition 4.} $\mathcal{C}_I$ for the Z-channel with $a_{21} \in (0,1)$ satisfies

- $\mathcal{C}_I \subseteq \mathcal{C}_{BC}(P_1 + P_2/a_{21}) \cap [0, C(P_1)] \times [0, C(P_2)]$
- $\left( C(P_1), C\left( \frac{P_2}{1 + a_{21}P_1} \right) \right)$ is achievable.
- $\max_{(R_1, R_2) \in \mathcal{C}_I} R_1 + R_2 = C(P_1) + C\left( \frac{P_2}{1 + a_{21}P_1} \right)$.

\textit{Proof.} Follows from the equivalence of degraded interference channel and Z-channel when $a_{21} \in (0,1)$.

\subsection{Proposition 5.} Consider the interference channel with $a_{21} \in (0,1)$ and $a_{12}$ arbitrary.

1. $\mathcal{C}_I \subseteq \mathcal{C}_{BC}(P_1 + P_2/a_{21}) \cap [0, C(P_1)] \times [0, C(P_2)]$
2. If $a_{12} \in (0,1)$, then $\mathcal{C}_I \subseteq \mathcal{C}_{BC}(P_1 + P_2/a_{21}) \cap \mathcal{C}_{BC}(P_2 + P_1/a_{12}) \cap [0, C(P_1)] \times [0, C(P_2)]$.

\textit{Proof.} Sufficient to prove (1). A genie provides $x^n_2$ and therefore $x^n_1 + \xi^n$ to decoder 1. Optimal processing will discard $x^n_2$, $y^n_1$ and operate on $x^n_1 + \xi^n$, the output of the Z-channel. So $\mathcal{C}_I \subseteq \mathcal{C}_{I,Z}(a_{21})$. (1) then follows from Prop. 4.

\section{Other results:}

- I. Sason’s achievable rate region: easier to compute subset of $\mathcal{F}^*(HK$ region).
- I. Sason’s shows that if $a_{12} = a_{21} \in (0,1)$, for high enough SNR, TDM/FDM is not optimal.
- Telatar and Tse: A new expression for converse with difference between $\mathcal{C}_{HK}$ and $\mathcal{C}_{outer}$ within 1 bit per received dimension (scalar case 1 bit/sample use).

Lecture 11: Gaussian interference channels, degraded interference channels, Z-channel-5