A Connections between superposition and Marton’s Theorem

- We may consider, as did Han – Kobayashi in their interference channel paper, superposition. This was Cover’s technique (1975), the basis for HK’s result.
- Consider $U_0, U_1, U_2$ independent. $U_0$ decoded by both. $U_1$ and $U_2$ by users 1 and 2 respectively.

$$\mathcal{P}_1 := \{ Z = U_{[3]}XY_{[2]} \}$$

and

$$\mathcal{P}_1^* := \{ Z = QU_{[3]}XY_{[2]}, \text{ with } U_0U_1U_2 \text{ conditionally independent given } Q \}$$

- Define

$$\mathcal{I}(Z) = \begin{cases} r_0 & \leq I(U_0; Y_1|U_1Q) \\ r_1 & \leq I(U_1; Y_1|U_0Q) \\ r_0 + r_1 & \leq I(U_0U_1; Y_1|Q) \\ r_0 & \leq I(U_0; Y_2|U_0Q) \\ r_2 & \leq I(U_2; Y_2|U_0Q) \\ r_0 + r_2 & \leq I(U_0U_2; Y_2|Q) \end{cases}$$

- Intersection of two polymatroids in 2 dimensions, extended to three dimensions in different directions. $\mathcal{I}^* := \text{cl conv } \bigcup_{Z \in \mathcal{P}_1} \mathcal{I}(Z)$ is achievable with common information.
- (Cover, 1975) Define $\mathcal{I} := \text{cl conv } \bigcup_{Z \in \mathcal{P}_1} \mathcal{I}(Z) = \text{cl conv } \mathcal{I}_0$.

Moreover

$$\mathcal{D} := \text{closure } \{(r_1 + s_1, r_2 + s_2) : (r_1, r_2, s_1 + s_2) \in \mathcal{I} \}$$

$$\mathcal{D}_0 := \text{closure conv } \{(r_1 + s_1, r_2 + s_2) : (r_1, r_2, s_1 + s_2) \in \mathcal{I}_0 \}$$

$$\mathcal{D}^* := \text{closure } \{(r_1 + s_1, r_2 + s_2) : (r_1, r_2, s_1 + s_2) \in \mathcal{I}^* \}$$

$$\mathcal{D}_0^* := \text{closure } \{(r_1 + s_1, r_2 + s_2) : (r_1, r_2, s_1 + s_2) \in \mathcal{I}_0^* \}$$

- Hajek–Pursley (1979) consider $\mathcal{D}_0$ and show that

$$\mathcal{D}_0 = \text{closure conv } \begin{cases} (R_1, R_2) & R_1 + R_2 \leq \min \{ I(U_0; Y_k), k = 1, 2 \} + I(U_1; Y_1|U_0) + I(U_2; Y_2|U_0) \\
& \text{such that } 1) U_{[3]} \rightarrow X \rightarrow Y_{[2]}, \\
& 2) U_{[3]} \text{ are independent.} \end{cases}$$

- Can easily extend equivalence of $\mathcal{D}^*$ with

$$\text{closure } \begin{cases} (R_1, R_2) \in \mathbb{R}_+^2 & R_1 + R_2 \leq \min \{ I(U_0; Y_k|Q), k = 1, 2 \} + I(U_1; Y_1|U_0Q) + I(U_2; Y_2|U_0Q) \\
& \text{for some } QU_{[3]}XY_{[2]} \text{ that satisfies} \\
& 1) \text{Given } Q, U_0U_1U_2 \text{ are mutually independent,} \\
& 2) QU_0U_1U_2 \rightarrow X \rightarrow Y_{[2]} \end{cases}$$
• With $Q\hat{U}_0$, it is easy to see that this region belongs to Marton’s region.

• Inclusion in the other direction, or otherwise is not known, i.e., given $Q$, $U_0U_1U_2$ may lead to a larger region. Marton shows $\mathcal{R} \supseteq \mathcal{R}_0$

• Is $\mathcal{R}_0 \subseteq \mathcal{R}$?

\[ \mathcal{R}_0 = \bigcup_{z \in \mathcal{R}} \mathcal{R}(Z), \mathcal{R} = \text{closure conv } \mathcal{R}_0. \]

• Clearly, since $\mathcal{R}_0 \subseteq \mathcal{R}$, we have $\mathcal{R}_0 \subseteq \mathcal{R}$.

• Let $(R_1, R_2) \in \{(r_1 + s_1, r_2 + s_2) : (r_1, r_2, s_1 + s_2) \in \mathcal{R}\}$.

\[ \implies \exists s_1 \geq 0, s_2 \geq 0 \text{ with } (r_1, r_2, s_1 + s_2) \in \mathcal{R}. \] If $(r_1, r_2, s_1 + s_2) \in \mathcal{R}_0$, then $(R_1, R_2) \in \mathcal{R}_0$ and we are done. If $(r_1, r_2, s_1 + s_2)$ is a limit point of conv $\mathcal{R}_0$, then an ε ball around this contains an $(r'_1, r'_2, s'_1 + s'_2) \in \text{conv } \mathcal{R}_0$, $(r'_1, r'_2, s'_1 + s'_2) \neq (r_1, r_2, s_1 + s_2)$, i.e.,

\[
(r'_1, r'_2, s'_1 + s'_2) = \sum_{\ell=1}^{L} \lambda_{\ell} \left( r'_1(\ell), r'_2(\ell), s'_1(\ell) + s'_2(\ell) \right). 
\]

If $(r'_1 + s'_1, r'_2 + s'_2) \neq (r_1 + s_1, r_2 + s_2)$, we will have shown $(r_1 + s_1, r_2 + s_2)$ is a limit point of the conv. $(r_1 + s_1, r_2 + s_2) : (r_1, r_2, s_1 + s_2) \in \mathcal{R}_0$, and hence in $\mathcal{R}_0$.

• Note that once a suitable $s'_1, s'_2$ is picked, since $s'_1 + s'_2$ is fixed and equals $\sum_{\ell=1}^{L} \lambda_{\ell}(s'_1(\ell) + s'_2(\ell))$, the individual $s'_1(\ell), s'_2(\ell)$ may be picked arbitrarily ($s'_1(\ell) \geq 0, s'_2(\ell) \geq 0, s'_1(\ell) + s'_2(\ell)$ is a given value).

• If $s_1 = s_2 = 0$.

- Suppose $s'_1 + s'_2 = 0$. Then $s'_1 = s'_2 = 0$ and since $(r'_1, r'_2, 0) \neq (r_1, r_2, 0)$, we must have $(r_1 + s_1, r_2 + s_2) \neq (r'_1 + s'_1, r'_2 + s'_2)$.

- Suppose $s'_1 + s'_2 > 0$. If $r_1 \neq r'_1$, choose $s'_1 = 0$; else choose $s'_2 = 0$. Then $(r_1 + s_1, r_2 + s_2) \neq (r'_1 + s'_1, r'_2 + s'_2)$.

• If $s_1 > 0, s_2 > 0$. Take $\epsilon$ sufficiently small so that $s_1 - \epsilon > 0, s_2 - \epsilon > 0$.

- If $r_1 \neq r'_1$, choose $s'_1 = s_1$; else choose $s'_2 = s_2$. (In this case $r_1 = r'_1$. If $s'_1 = s_2$, then we must have $r'_2 = r_2$). This ensures $(r'_1 + s'_1, r'_2 + s'_2) \neq (r_1 + s_1, r_2 + s_2)$.

• If $s_1 = 0, s_2 > 0$.

- If $r'_1 \neq r_1$, choose $s'_1 = 0$.

- If $r'_1 = r_1$, Suppose $r'_2 - r_2 \neq s_1 + s_2 - s'_1 - s'_2$. Then set $s'_1 = 0$. Then too $r'_2 + s'_2 \neq r_2 + s_2$.

Lecture 15 : Connections between Superposition and Marton’s Theorem-2
- If $r_1' = r_1$ and $r_2' - r_2 = s_2 - s_1' - s'_2$ (under $s_1 = 0, s_2 > 0$). Choose $s_1' = \frac{|s_1 + s_2 - (s_1' + s'_2)|}{2} \leq \epsilon$.

Then set $s_2' + \frac{|s_1 + s_2 - (s_1' + s'_2)|}{2} = s_2' + s_1' \geq s_2 + s_1' - \epsilon$

$\implies s_2' \geq 0, |s_2 - s_2'| \leq 3\epsilon/2.$

Thus $d\left((r_1 + s_1, r_2 + s_2), (r_1' + s_1', r_2' + s_2')\right) \leq \sqrt{\frac{3\epsilon^2}{4} \times 2} = \frac{3}{\sqrt{2}}\epsilon$ (and $(r_1 + s_1, r_2 + s_2) \neq (r_1' + s_1', r_2' + s_2')$). Thus $\mathcal{D} = \mathcal{D}_0.$