1 Network preliminaries

Definition 1 (Directed Graph). \((N, E, c)\) is a directed graph consisting of a set of nodes \(N\), the directed edge set \(E \subset N \times N\) (an edge is an ordered pair \((i, j)\) where \(i, j \in N\)) and the capacity function \(c : E \to \mathbb{R}_+\).

Definition 2 (Path). Path of length \(k\) from \(i \in N\) to \(j \in N\) is the sequence of nodes \(P = i = i_0, i_1, i_2, \ldots, i_{k-1}, i_k = j\) such that
- \(i_1, i_2, \ldots, i_{k-1}\) are distinct, different from \(i\) and \(j\)
- \(i_m, i_{m+1} \in E\).

Definition 3 (Cycle). A path with \(i_0 = i_k\).

Definition 4 (Cut). \((M, N \setminus M)\) where \(M \subseteq N\).

Definition 5 \((c(M, L))\). Suppose \(M, L \subseteq N\). Define
\[
c(M, L) := \sum_{ij \in M \cup L} c_{ij}
\]

Definition 6 \((\text{net } c(i))\).
\[
\text{net } c(i) := c(\{i\}, N) - c(N, \{i\})
\]
- outflow including self loop \((i, i)\) – inflow including self loop \((i, i)\)
- = net outflow from \(i\).

\*\([J] \subseteq N\) represents supply nodes; the first \(J\) nodes.
\*\(t_0 \in N \setminus [J]\) represents sink node.
\*\(N \setminus ([J] \cup \{t_0\})\) intermediate nodes.

Assumptions
1) For each \(j \in [J]\), there is at least one path from \(j\) to \(t_0\).
2) No cycles.

Definition 7 (Feasible flow). \(f : E \to \mathbb{R}_+\) such that
1) \(\forall ij \in E, 0 \leq f_{ij} \leq c_{ij}\)
2)
\[
\text{net}(f, k) \geq 0, \text{ if } k \in [J]
\]
\[
\text{net}(f, k) \leq 0, \text{ if } k = t_0
\]
\[
\text{net}(f, k) = 0, \text{ otherwise}
\]

Lemma 1. (Meggido, 1974) Let \(\rho\) be the min-cut capacity function defined on subsets of \([J]\) as follows.
\[
\rho(S) := \min \{c(M, N \setminus M) : S \subseteq M, t_0 \in N - M, M \subseteq N\}.
\]
- Then \(\rho\) is a sub-modular rank function.
• If \( R(S) \leq \rho(S), \forall S \subseteq \Sigma_0 \), then there is a flow such that \( R_s = \text{net}(f, S), \forall S \in [J] \). If there exists a flow such that \( R_s = \text{net}(f, S), \forall S \in [J] \), then \( R(S) \leq \rho(S), \forall S \subseteq [J] \).

**Remark 1.** Generalization of max flow min cut theorem for multiple source nodes, single destination.

**Remark 2.** Any flow is a sum of path flows. In a path \( P_{sk} \), flow is \( R_{sk} \) on all edges.

**Lemma 2.** (Ford–Fulkerson) For any flow \( f \), there exist paths \( P_{sk} (s \in [J], k \in [m_s]) \), from \( s \) to \( t_0 \) and associated nonnegative real numbers \( R_{sk} (s \in [J], k \in [m_s]) \) such that

1) \( \text{net}(f, s) = \sum_{k=1}^{m_s} R_{sk}, \forall s \in [J] \)

2) \( \sum_{P_{sk} : ij \in P_{sk}} R_{sk} \leq c_{ij}, \forall ij \in B \)