A Tight Rate Bound and Matching Construction for Locally Recoverable Codes with Sequential Recovery

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Information Theory and Applications Workshop
Catamaran Resort, San Diego, February 13, 2017

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Thanks to Alon and the other organizers for the invite...

Parameters of Interest

All codes are linear codes over $\mathbb{F}_q$.

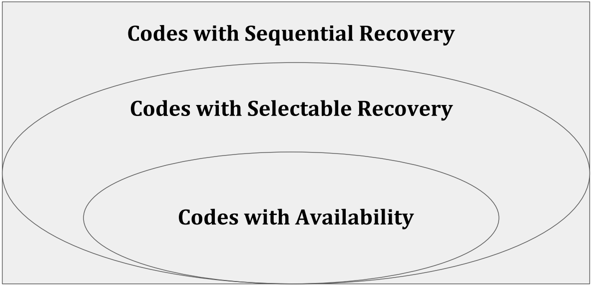
q	field size
n	Block length
k	dimension
R	$= (k/n) =$ rate
$d_{\min}$	minimum distance

t	maximum number of erasures from which local recovery is desired
r	maximum number of code symbols contacted for recovery of a single erased symbol

Various Local Approaches to Multiple-Erasure Recovery

Approach	Explanation
Availability	Each code symbol is protected by a set of parity checks that includes a set of t orthogonal parity checks, each of weight $\leq (r+1)$
Sequential Recovery	Given a set of $\leq t$ erased code symbols, there exists a parity check for at least one code symbol, of weight $\leq (r+1)$ which does not include any of the other code symbols
Selectable Recovery	For any given set of $(t-1)$ other erasures, each code symbol is protected by a parity check of weight $\leq (r+1)$, that does not include any of these other code symbols
Cooperative Recovery	Given a set of t erasures, there exists a set of r code symbols using which one can recover from these t erasures

Codes with Locality For Multiple Erasures



Sequential Recovery Results

- A tight upper bound on rate R for any $(r \geq 3,\ t \geq 2)$
- Matching binary-code construction

Codes with Locality for Sequential Recovery

Definition

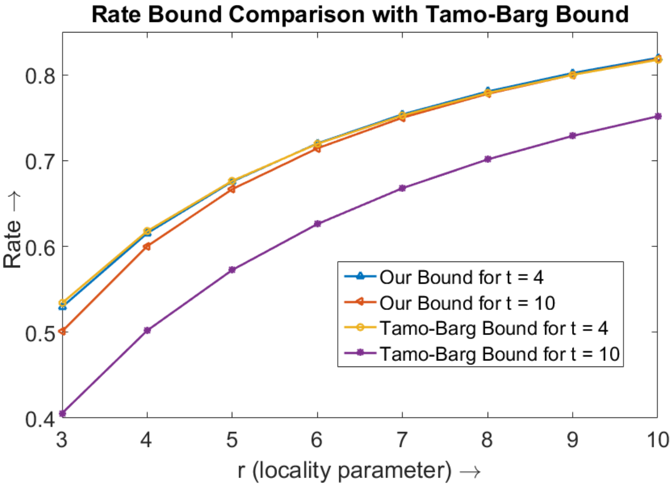
An $[n, k]$ code over $\mathbb{F}_q$ is a

code with sequential recovery from t erasures having locality r

if for any set of $s \leq t$ erased symbols, $\{c_{\sigma_1}, ..., c_{\sigma_s}\}$, there exists a codeword $\underline{h}$ in the dual code, of Hamming weight $\leq r + 1$, such that $\text{supp}(\underline{h}) \cap \{\sigma_1, ..., \sigma_s\} = 1$.

We denote the above defined codes as $(n, k, r, t)_{seq}$ codes.

Motivation:
Improved Rate (in comparison with availability)



The (Tight) Upper Bound on Rate

Theorem

Rate Bound: *Let $\mathcal{C}$ be an $(n, k, r, t)_{seq}$ code over a field $\mathbb{F}_q$. Let $r \geq 3$. Then*

$$\frac{k}{n} \leq \frac{r^{\frac{t}{2}}}{r^{\frac{t}{2}} + 2\sum_{i=0}^{\frac{t}{2}-1} r^i} \quad \textit{for } t \textit{ an even integer,} \tag{1}$$

$$\frac{k}{n} \leq \frac{r^s}{r^s + 2\sum_{i=1}^{s-1} r^i + 1} \quad \textit{for } t \textit{ an odd integer,} \tag{2}$$

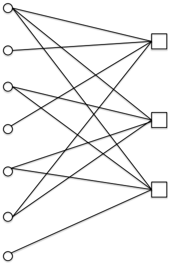
where $s = \frac{t+1}{2}$.

Bound for $t = 2, 3, 4, 5, 6, 7$

t	Bound	t	Bound
2	$\frac{r}{r+2}$	3	$\frac{r^2}{r^2+2r+1}$
4	$\frac{r^2}{r^2+2r+2}$	5	$\frac{r^3}{r^3+2r^2+2r+1}$
6	$\frac{r^3}{r^3+2r^2+2r+2}$	7	$\frac{r^4}{r^4+2r^3+2r^2+2r+1}$

Rawat et. al. Construction for Codes with Sequential Recovery

1. r+1-regular bipartite graph, girth $\geq t+1$ [a]
2. edge are code symbols, node are parity checks,
3. Rate is $\frac{r-1}{r+1} + \frac{1}{n}$.
4. Rate meets our rate bound when a Moore graph of degree $r+1$ and girth $t+1$ exists.
5. Such Moore graphs are shown to not exist for $t \notin \{2,3,4,5,7,11\}$ for any $r \geq 2$.
6. Thus the construction is not rate-optimal in most cases.



[a] A. Rawat, A. Mazumdar, and S. Vishwanath, "On cooperative local repair in distributed storage, " in Information Sciences and Systems (CISS), 2014 48th Annual Conference on, 2014, pp. 1?5.

Conjecture on Rate

The following conjecture on the rate of an $(n, k, r, t)_{seq}$ code was given by Song[2] et al.

$$\frac{k}{n} \leq \frac{1}{1+\sum_{i=1}^{m} \frac{a_i}{r^i}},$$
$$a_i \geq 0, \quad a_i \in \mathbb{Z}, \quad \sum_{i=1}^{m} a_i = t, \quad m = \lceil log_r(k) \rceil.$$

- Our rate bound verifies this general conjecture,

- More specific bounds were conjectured for $t = 5, 6$
- The $t = 5$ bound was verified to be correct, the $t = 6$ case, the conjecture turned out to be incorrect .

[2] Song, Cai, Yuen, "On sequential locally repairable codes," arXiv preprint arXiv:1610.09767, 2016.

Proof (restrict to a sub matrix of H_{full})

Let H_{full} be the parity-check matrix of the $[n, k, r, t]_{\text{seq}}$ code.

$$H_{\text{full}} \;\Rightarrow\; \text{Rowspace}\,(H_{\text{full}})$$

$$\Downarrow$$

$$\text{restrict to subspace } S \text{ spanned by rows of } w_H(\cdot) \leq (r+1)$$

$$\Downarrow$$

$$\text{select basis with } w_H(\cdot) \leq (r+1) \text{ for } S\text{: } \{\underline{h}_1^t, \underline{h}_2^t, \cdots \underline{h}_m^t\}$$

$$\Downarrow$$

$$\text{Set } H \;=\; \begin{bmatrix} \underline{h}_1^t \\ \underline{h}_2^t \\ \vdots \\ \underline{h}_m^t \end{bmatrix}.$$

Proof (choose columns of low Hamming weight)

$$H = \begin{bmatrix} \underline{h}_1^t \\ \underline{h}_2^t \\ \vdots \\ \underline{h}_m^t \end{bmatrix} \Rightarrow \text{(each row of } H \text{ has Hamming weight} \leq (r+1))$$

- Our interest is in maximizing rate
- i.e., maximizing n for given redundancy m
- Hence choose as many columns of low Hamming weight as possible

Proof
(after some work, for t even, arrive at form below)

$$
H = \left[\begin{array}{c|c|c|c|c|c|c|c|c}
D_0 & A_1 & 0 & 0 & \dots & 0 & 0 & 0 & \\ \cline{1-8}
0 & D_1 & A_2 & 0 & \dots & 0 & 0 & 0 & \\ \cline{1-8}
0 & 0 & D_2 & A_3 & \dots & 0 & 0 & 0 & \\ \cline{1-8}
0 & 0 & 0 & D_3 & \dots & 0 & 0 & 0 & \\ \cline{1-8}
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & E \\ \cline{1-8}
0 & 0 & 0 & 0 & \dots & A_{\frac{t}{2}-2} & 0 & 0 & \\ \cline{1-8}
0 & 0 & 0 & 0 & \dots & D_{\frac{t}{2}-2} & A_{\frac{t}{2}-1} & 0 & \\ \cline{1-8}
0 & 0 & 0 & 0 & \dots & 0 & D_{\frac{t}{2}-1} & & \\ \cline{1-7}
0 & 0 & 0 & 0 & \dots & 0 & 0 & C &
\end{array}\right]
$$

- $\{D_i\}$ are diagonal and $\{A_i\}$ have column weight 1
- each row of $\{A_i\}$ has Hamming weight $\leq r$
- $\{C\}$ has column weight 2 and $\{E\}$ has column weight ≥ 3

- this form leads to the bound

Proof
(if equality holds in the bound, we must have:)

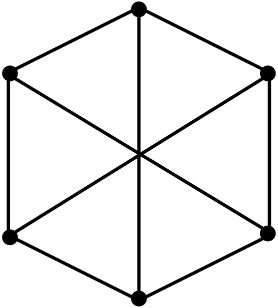
$$H = \left[\begin{array}{c|c|c|c|c|c|c|c}
D_0 & A_1 & 0 & 0 & \dots & 0 & 0 & 0 \\ \hline
0 & D_1 & A_2 & 0 & \dots & 0 & 0 & 0 \\ \hline
0 & 0 & D_2 & A_3 & \dots & 0 & 0 & 0 \\ \hline
0 & 0 & 0 & D_3 & \dots & 0 & 0 & 0 \\ \hline
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \hline
0 & 0 & 0 & 0 & \dots & A_{\frac{t}{2}-2} & 0 & 0 \\ \hline
0 & 0 & 0 & 0 & \dots & D_{\frac{t}{2}-2} & A_{\frac{t}{2}-1} & 0 \\ \hline
0 & 0 & 0 & 0 & \dots & 0 & D_{\frac{t}{2}-1} & C
\end{array}\right]$$

- each row of $\{A_i\}$ has Hamming weight $= r$
- there is no matrix E

Example: PC Matrix of Rate-Optimal ($r = 3, t = 2$) Code (Regular Graph)

$$
\begin{aligned}
H &= [D_0 \mid A_1] \\
&= \left[\begin{array}{cccccc|ccccccccc}
1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1
\end{array}\right]
\end{aligned}
$$

Regular Graph Interpretation of $(r = 3, t = 2)$ Rate-Optimal Code



- Edges correspond to symbols
- Nodes correspond to parity-symbols
- Thus this code is an $[n = 15,\ k = 9,\ r = 3,\ t = 2]$ code

- A regular graph for any r will lead to a rate-optimal $(t = 2)$ code

Rate-Optimal Product Code ($t = 3$, $r = 4$)

c_{11}	c_{12}	c_{13}	c_{14}	c_{15}
c_{21}	c_{22}	c_{23}	c_{24}	c_{25}
c_{31}	c_{32}	c_{33}	c_{34}	c_{35}
c_{41}	c_{42}	c_{43}	c_{44}	c_{45}
c_{51}	c_{52}	c_{53}	c_{54}	c_{55}

Figure: Code array.

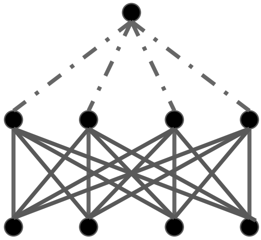


Figure: Graphical representation.

- Thus this code is an $[n = 25,\ k = 16,\ r = 4,\ t = 3]$ code
- A $2D$ product code for any r will lead to a rate-optimal ($t = 3$) code

Product Code: Parity Check Matrix

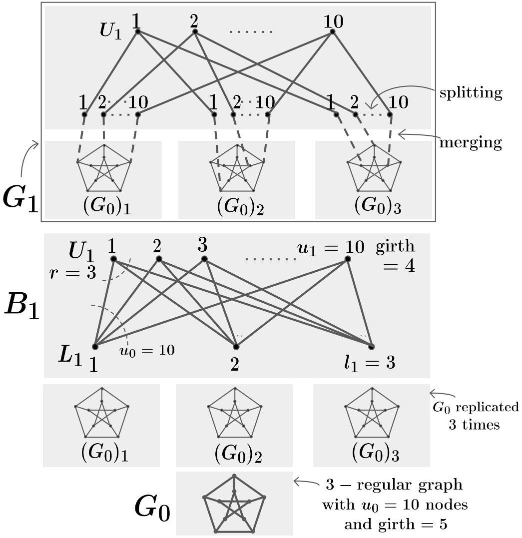
$$H = \left[\begin{array}{ccccc|cccccccccccccccccccc}
1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
\hline
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1
\end{array}\right]$$

(this matrix also conforms to the form

$$H = \begin{bmatrix} D_0 & A_1 \\ 0 & D_1 \end{bmatrix},$$

in the proof of the theorem)

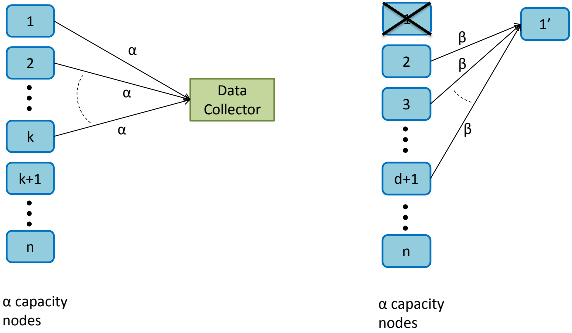
A Binary Rate-Optimal Code ($t = 4, r = 3$)



Regenerating Codes

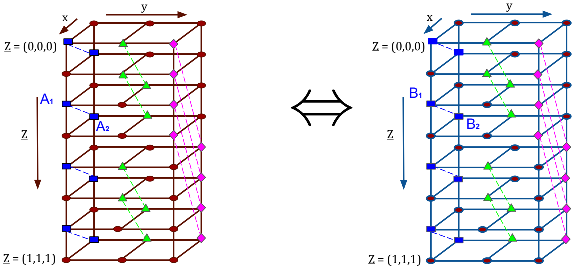
Regenerating Codes - Formal Definition

Parameters: $(\ (n, k, d), \ (\alpha, \beta), \ B, \ \mathbb{F}_q \ )$



- Data to be recovered by connecting to any k of n nodes
- Nodes to be repaired by connecting to any d nodes, downloading β symbols from each node; ($d\beta <<$ file size B)

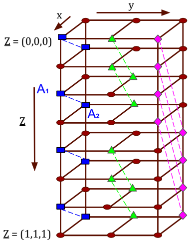
A Recent High-Rate MSR Code Construction with $d < (n-1)$



- A simple interpretation as being obtained through a pairwise, symbol transformation from a layered RS code (left)
- ... followed by an RS code across symbols within a node
- extends the Ye-Barg construction...

- Min Ye, Alexander Barg, "Explicit constructions of optimal-access MDS codes with nearly optimal sub-packetization," arXiv:1605.08630v1, 27 May 2016.
- Birenjith Sasidharan, Myna Vajha, P. Vijay Kumar, "An Explicit, Coupled-Layer Construction of a High-Rate MSR Code with Low Sub-Packetization Level, Small Field Size and $d < (n-1)$," arXiv:1701.07447v1, 25 Jan 2017.

A Recent High-Rate MSR Code Construction with $d < (n-1)$



- Rate as close to 1 as desired
- Field size comparable to that of an RS code of same block length
- Sub-packetization level comparable to that of the Ye-Barg construction
- $d < (n-1)$
- Explicit with uniform, all-symbol repair

- Birenjith Sasidharan, Myna Vajha, P. Vijay Kumar, "An Explicit, Coupled-Layer Construction of a High-Rate MSR Code with Low Sub-Packetization Level, Small Field Size and $d < (n-1)$," arXiv:1701.07447v1, 25 Jan 2017.

Thanks!